ECON 203
Midterm on Consumer Theory

Be sure to show your work for all answers, even if the work is simple.

This exam will begin at 17:40 and end at 19:20

1. *(15 points)* **Honor Statement:** Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test. I will also not use a calculator or other electronic aid for calculation during this test.

   Name and Surname: _
   Student ID: _
   Signature:
   _

2. *(21 points total)* Let $h_x(p_x, p_y, u)$ be the Hicksian or income compensated demand for x, $I(p_x, p_y, u)$ the amount of income necessary for a consumer to achieve a utility level of u, and $X(p_x, p_y, I)$ be their normal or Marshallian demand for x. You may assume that $\frac{\partial I}{\partial p_x} = x$, and $h_x(p_x, p_y, u) = X(p_x, p_y, I(p_x, p_y, u))$.

   (a) *(8 points)* Derive the Slutsky equation in elasticity form, defining each term as you go.

   Solution 1 *From the Duality theorem we know:*

   $$h_x(p_x, p_y, u) = X(p_x, p_y, I(p_x, p_y, u))$$

   If we take the derivative of this equation with regard to p_x *we get:*

   $$\frac{\partial h_x}{\partial p_x} = \frac{\partial X}{\partial p_x} + \frac{\partial X}{\partial I}\frac{\partial I}{\partial p_x}$$

   from the Envelope theorem we know $\frac{\partial I}{\partial p_x} = X$*, thus we have:*

   $$\frac{\partial h_x}{\partial p_x} = \frac{\partial X}{\partial p_x} + \frac{\partial X}{\partial I}X$$

   To convert the first two terms into elasiticities we multiply the whole equation by p_x/X *and then do the conversion:*

   $$\begin{aligned} \frac{\partial h_x}{\partial p_x}\frac{p_x}{X} &= \frac{\partial X}{\partial p_x}\frac{p_x}{X} + \frac{\partial X}{\partial I}\frac{p_x}{X}X \\ e_{h_x}(p_x) &= e_x(p_x) + \frac{\partial X}{\partial I}\frac{1}{X}p_x X \end{aligned}$$

where $e_{h_x}(p_x)$ *is the Hicksian own price elasticity, and* $e_x(p_x)$ *is the normal one. To get the last term into an elasticity, we need to multiply by* I*, we also divide by* I *so that the equation does not change:*

$$\begin{aligned} e_{h_x}(p_x) &= e_x(p_x) + \frac{\partial X}{\partial I}\frac{I}{X}\frac{p_x X}{I} \\ e_{h_x}(p_x) &= e_x(p_x) + e_x(I)\, s_x \end{aligned}$$

where $e_x(I)$ *is the income elasticity with regards to* x*, and* $s_x = \frac{p_x X}{I}$ *is the share of income spent on* x*. To write it as the Slutsky equation we then move* $e_x(I)\, s_x$ *to the right hand side, or*

$$e_x(p_x) = e_{h_x}(p_x) - e_x(I)\, s_x$$

(b) *(3 points)* Define a Giffen good.

Solution 2 *A* Giffen good *is a good with an upward sloping demand function. OR a rise in price causes a rise in quantity demanded, or* $e_x(p_x) > 0$*, or* $\frac{\partial X}{\partial p_x} > 0$.

(c) *(6 points)* Using the Slutsky equation explain how a good can be a Giffen good, and why the search for these goods have primarily focused on staple goods—a type of food that constitutes most of what the very poor eat. What is the only statistically verified Giffen good found to date?

Solution 3 *For clarity we write the equation again:*

$$e_x(p_x) = e_{h_x}(p_x) - e_x(I)\, s_x$$

if $e_x(p_x) > 0$ *we need* $e_{h_x}(p_x) - e_x(I)\, s_x > 0$*, since* $e_{h_x}(p_x) < 0$ *we need* $-e_x(I)\, s_x > -e_{h_x}(p_x)$*. For this to be true we need:*

i. $e_{h_x}(p_x)$ *to be "small" or the good has few substitutes.*

ii. $e_x(I) < 0$ *the good must be inferior, and significantly so*

iii. s_x *needs to be large*

Staple foods are obviously the best candidates for all of these categories. Because it is the core food people eat it has few substitutes, generally staples are inferior, and probably it has the largest income share of any good.

Indeed the only statistically verified Giffen good is rice in the Hunan province of China.

(d) *(4 points)* A government wants to increase the health of their people, and are considering subsidizing the staple food, but they know that at least for some of the people this staple food is a Giffen good. Argue both that regardless of this they should subsidize the staple, and that because of this they should not subsidize the staple. What is the key factor that determines whether they should subsidize the staple or not?

Solution 4 *As I expected, most of you did not figure this out. Indeed only one student did. The key question is what they are going to substitute for. If they will substitute the staple food with alcohol (for example) this will not improve their health. If they are substituting it for a* small *amount of meat, it probably will.*

This is something that can be empirically verified. Simply have your health experts conduct a survey based of wealth and diet. It probably will not be strictly increasing, and this will make great popular news items, but over the critical range? If it is monotonic increasing then the Giffen good consumers are not an issue. If it is not then, well, care needs to be taken and the subsidy might not be a good idea.

α	β	$F(p_f, p_c, C)$	$C(p_c, p_f, I)$	$F(p_c, p_f, I)$	s_f
$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{8}C\frac{p_c}{p_f}$	$\frac{8}{11}\frac{I}{p_c}$	$\frac{3}{11}\frac{I}{p_f}$	$\frac{3}{11}$
$\frac{1}{2}$	$\frac{3}{5}$	$\frac{5}{6}C\frac{p_c}{p_f}$	$\frac{6}{11}\frac{I}{p_c}$	$\frac{5}{11}\frac{I}{p_f}$	$\frac{5}{11}$
$\frac{3}{4}$	$\frac{1}{3}$	$\frac{9}{4}C\frac{p_c}{p_f}$	$\frac{4}{13}\frac{I}{p_c}$	$\frac{9}{13}\frac{I}{p_f}$	$\frac{9}{13}$
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{2}C\frac{p_c}{p_f}$	$\frac{2}{3}\frac{I}{p_c}$	$\frac{1}{3}\frac{I}{p_f}$	$\frac{1}{3}$

3. *(23 points total)* Consider the utility function $U(F, C) = F^{\alpha}C^{\beta}$.

   (a) *(2 points)* Set up the Lagrangian you would use to find the optimal consumptions over the budget set $p_f F + p_c C \leq I$.

   Solution 5

$$L(F, C, \lambda) = F^{\alpha}C^{\beta} - \lambda(p_f F + p_c C - I)$$

   (b) *(4 points)* Find the first order conditions of this objective function.

   Solution 6

$$\begin{aligned} \frac{\partial L}{\partial F} &= \alpha\frac{U}{F} - \lambda p_f = 0 \\ \frac{\partial L}{\partial C} &= \beta\frac{U}{C} - \lambda p_c = 0 \\ \frac{\partial L}{\partial \lambda} &= -(p_f F + p_c C - I) = 0 \end{aligned}$$

Remark 7 *OK, people. You completely ignored my advice about the derivative of a Cobb-Douglass. And so many of you—because of this—got the wrong answers. This is the* **second time** *I have asked this question.*

Of course some of you just don't know how to take derivatives. You should change your major, you are an embarrassment on the department.

(c) *(2 points)* Solve for a function for F in terms of C and prices.

Solution 8 *From* $\alpha\frac{U}{F} - \lambda p_f$ *we know that* $\lambda = \frac{1}{F}U\frac{\alpha}{p_f}$*, from* $\frac{\partial L}{\partial C}$ *we know that* $\lambda = \frac{1}{C}U\frac{\beta}{p_c}$ *thus*

$$\frac{1}{F}U\frac{\alpha}{p_f} = \frac{1}{C}U\frac{\beta}{p_c} \Rightarrow F = C\frac{\alpha}{\beta}\frac{p_c}{p_f}$$

or we could have proceeded by remembering that:

$$\begin{aligned} BfB_F &= \frac{1}{F}U\frac{\alpha}{p_f} \\ BfB_C &= \frac{1}{C}U\frac{\beta}{p_c} \end{aligned}$$

and that they should be equal.

(d) *(2 points)* Find the demand for C.

Solution 9 *If you got this wrong because you don't know what a demand curve is you should be so ashamed of yourself that you quit this class.*

A demand curve is a function from the exogenous (p_f, p_c, I) *to the* endogenous*, or choice variables. This is the amount of* C *you consume. Anyway, to solve this:*

$$\begin{aligned} F &= C\frac{\alpha}{\beta}\frac{p_c}{p_f} \\ (p_f F + p_c C - I) &= 0 \end{aligned}$$

$$\begin{aligned} p_f C\frac{\alpha}{\beta}\frac{p_c}{p_f} + p_c C &= I \\ \frac{C}{\beta}p_c(\alpha+\beta) &= I \\ C &= \frac{\beta}{\alpha+\beta}\frac{I}{p_c} \end{aligned}$$

(e) *(5 points)* Find the demand for F using two different methods and verify that your answer is correct.

Solution 10 *The reason I asked for both is because it is* ***very*** *important that you realize that with most math problems you can doulbe check your answers. You should do this as a matter of course, here I am giving you points for it. The two methods are using the budget constraint and equalizing the bang for the bucks.*

Budget constraint:

$$\begin{aligned}p_f F + p_c C &= I \\ C &= \frac{\beta}{\alpha+\beta}\frac{I}{p_c}\end{aligned}$$

$$\begin{aligned}p_f F + p_c\left(\frac{\beta}{\alpha+\beta}\frac{I}{p_c}\right) &= I \\ p_f F + \beta\frac{I}{\alpha+\beta} &= I\end{aligned}$$

$$\begin{aligned}p_f F &= I - \beta\frac{I}{\alpha+\beta} \\ p_f F &= \alpha\frac{I}{\alpha+\beta} \\ F &= \frac{\alpha}{\alpha+\beta}\frac{I}{p_f}\end{aligned}$$

or we could use equalizing the BfB's:

$$\begin{aligned}\frac{1}{F}U\frac{\alpha}{p_f} &= \frac{1}{C}U\frac{\beta}{p_c} \\ C &= \frac{\beta}{\alpha+\beta}\frac{I}{p_c}\end{aligned}$$

$$\begin{aligned}\frac{1}{F}U\frac{\alpha}{p_f} &= \frac{1}{\left(\frac{\beta}{\alpha+\beta}\frac{I}{p_c}\right)}U\frac{\beta}{p_c} \\ \frac{1}{F}U\frac{\alpha}{p_f} &= \frac{U}{I}(\alpha+\beta) \\ \alpha\frac{I}{p_f(\alpha+\beta)} &= F\end{aligned}$$

(f) *(8 points)* Find the elasticity of F with respect to p_c and I, and the share of income that is spent on food. Why the elasticity of food with respect to income a problem?

Solution 11 *Like always we need to first find the derivatives, since*

$$F = \frac{\alpha}{\alpha+\beta}\frac{I}{p_f}$$

we see that $\frac{\partial F}{\partial p_c} = 0$ *and* $\frac{\partial F}{\partial I} = \frac{\alpha}{\alpha+\beta}\frac{1}{p_f}$ *from this it is clear that:*

$$e_F(p_c) = \frac{\partial F}{\partial p_c}\frac{p_c}{F} = 0$$

and we now realize that:

$$\begin{aligned} e_F(I) &= \frac{\partial F}{\partial I}\frac{I}{F} = \frac{\alpha}{\alpha+\beta}\frac{1}{p_f}\frac{I}{F} \\ &= \frac{\alpha}{\alpha+\beta}\frac{I}{p_f}\frac{1}{F} \end{aligned}$$

But since $F = \frac{\alpha}{\alpha+\beta}\frac{I}{p_f}$ *we have:*

$$e_F(I) = \frac{\alpha}{\alpha+\beta}\frac{I}{p_f}\frac{1}{F} = \frac{F}{F} = 1$$

And finally by definition:

$$s_f = \frac{p_f F}{I} = \frac{p_f\left(\frac{\alpha}{\alpha+\beta}\frac{I}{p_f}\right)}{I} = \frac{\alpha}{\alpha+\beta}$$

many of you remembered that $e_F(I) = 1$ *is a problem because Engel's Law is that* $0 < e_F(I) < 1$.

4. *(15 points total)* About *rationality*

   (a) *(3 points)* What is the motivation for Economists (as social scientists) to assume rationality?

   Solution 12 *Do not assume people are stupid.*

   (b) *(9 points)* Write down and define the three axioms that are required for the normative definition of rationality.

   Solution 13 *As I told anyone who asked, you could use either math or words for your definitions, and some are easier to define using words.*

   i. *Reflexivity:* $A \succsim A$, *or* A *is always at least as good as* A
   ii. *Transitivity:* $A \succsim B$ *and* $B \succsim C$ *means that* $A \succsim C$, *or the person has no decision cycles.*
   iii. *Completeness: For all* A *and* B, $A \succsim B$, $B \succsim A$ *or both, or the person can compare all options.*

   (c) *(3 points)* For one of the three axioms you just defined, show what indifference curves would look like if they did not satisfy that axiom.

   Solution 14 *If you gave two answers I only graded your first answer, this didn't cost anyone though. I am too lazy to draw the graphs.*

   Transitivity: any crossing indifference curves would do. This is ruled about by this axiom.

Completeness: any set of non-nested indifference curves will do.

μ	τ	$F = \frac{I}{p_f}$ *if*
2	7	$\frac{2}{7}p_c > p_f$
8	3	$\frac{8}{3}p_c > p_f$
1	5	$\frac{1}{5}p_c > p_f$
4	6	$\frac{2}{3}p_c > p_f$

5. *(26 points total)* Consider the utility function $U(F,C) = \mu F + \tau C$.

(a) *(6 points)* Find the marginal utilities of food and clothing, is this utility function (strongly) monotonic? Is it convex?

Solution 15

$$MU_F = \mu,\ MU_C = \tau$$

since these are both strictly positive this function is (strongly) montonic. The question of whether it is convex is harder, the standard answer is

$$MRS = \frac{MU_C}{MU_F} = \frac{\tau}{\beta}$$

is non-increasing in F so this is true. However many of you thought since it was not strictly *decreasing in F this was a contradiction. For those and just for the fun of it let me show you another way:*

$$\begin{aligned} U(F,C) &= \mu F + \tau C \\ U\left(\tilde{F},\tilde{C}\right) &= \mu\tilde{F} + \tau\tilde{C} \end{aligned}$$

$$\begin{aligned} U\left(\lambda F + (1-\lambda)\tilde{F}, \lambda C + (1-\lambda)\tilde{C}\right) &= \mu\left(\lambda F + (1-\lambda)\tilde{F}\right) + \tau\left(\lambda C + (1-\lambda)\tilde{C}\right) \\ &= \lambda\mu F + (1-\lambda)\mu\tilde{F} + \lambda\tau C + (1-\lambda)\tau\tilde{C} \\ &= \lambda\mu F + \lambda\tau C + (1-\lambda)\mu\tilde{F} + (1-\lambda)\tau\tilde{C} \\ &= \lambda(\mu F + \tau C) + (1-\lambda)\left(\mu\tilde{F} + \tau\tilde{C}\right) \end{aligned}$$

$$U\left(\lambda F + (1-\lambda)\tilde{F}, \lambda C + (1-\lambda)\tilde{C}\right) = \lambda U(F,C) + (1-\lambda)U\left(\tilde{F},\tilde{C}\right)$$

so anything that is lower than both $U(F,C)$ and $U\left(\tilde{F},\tilde{C}\right)$ will also be lower than $U\left(\lambda F + (1-\lambda)\tilde{F}, \lambda C + (1-\lambda)\tilde{C}\right)$ for any $\lambda \in [0,1]$ and these preferences are convex.

(b) *(2 points)* Set up the Lagrangian for maximizing this over the budget set $p_f F + p_c C \leq I$.

$$L(F,C,\lambda) = \mu F + \tau C - \lambda(p_f F + p_c C - I)$$

(c) *(6 points)* Find the first derivatives of this objective function. Which of these are you certain must be equal to zero in any optimum? Why?

Solution 16 *The derivatives are:*

$$\begin{aligned}\frac{\partial L}{\partial F} &= \mu - \lambda p_f \\ \frac{\partial L}{\partial C} &= \tau - \lambda p_c \\ \frac{\partial L}{\partial \lambda} &= -\left(p_f F + p_c C - I\right)\end{aligned}$$

and those who glibly answered "but professor we ***know*** *they are* ***always all*** *equal to zero" you are wrong, obviously do not understand economics or math, and can't even realize when the professor is dropping a hint. I would not have asked this question if that was the answer. The answer of course is:*

Since this utility funcition is strongly montonic we know that we will spend all our income thus $\frac{\partial L}{\partial \lambda} = 0$.

To be more precise if (as many of you do) we solve the first equation for λ*, then* $\lambda = \frac{\mu}{p_f}$*, if we solve the second then* $\lambda = \frac{\tau}{p_c}$*. Given that you understand what we are doing,* $\frac{\mu}{p_f} = \frac{\tau}{p_c}$ *only by coincidence so we can not have both be true. The correct answer is that* $\lambda = \max\left\{\frac{\mu}{p_f}, \frac{\tau}{p_c}\right\} = \max\left\{BfB_F, BfB_C\right\}$ *and if* $BfB_F > BfB_C$ *that means* $C^* = 0, F^* = \frac{I}{p_f}$

(d) *(4 points)* Define a *corner solution.* In the real world are corner solutions common or rare? Explain your answer.

Solution 17 *A corner solution is when one or more of our choice variables is zero. I.e. this customer demands nothing of one or more goods.*

These are extremely common in the real world. In fact I do not think there is anyone who consumes all available goods.

The product space is simply too dense, too diverse. This does not rely on something I do not consume for religious or personal reasons (like vegetarianism). There are many meats, for example, that I would eat if I had no choice but I never choose to eat them.

(e) *(4 points)* Show that for almost all (p_f, p_c) this consumer will be at a corner solution.

Solution 18 *Again, reading this question should make you realize that in part c not ALL the first derivatives can be equal to zero. '*

Indeed one solution is: "If $\frac{\partial L}{\partial F} = \frac{\partial L}{\partial C} = 0$ *then* $\frac{\mu}{p_f} = \frac{\tau}{p_c}$*, which will not happen in general thus the consumer will be at a corner."*

Another solution is to recognize that: $BfB_F = \frac{\mu}{p_f}$ *and* $BfB_C = \frac{\tau}{p_c}$*, and if* $\frac{\mu}{p_f} > \frac{\tau}{p_c}$ *we should consume all* F *and no* C*. If the reverese is true then we consume only* C*, and we might consume both when* $\frac{\mu}{p_f} = \frac{\tau}{p_c}$*.*
A final solution is since $p_f F + p_c C - I = 0$*, we know that* $C = \frac{I}{p_c} - \frac{F}{p_c}p_f$*, substituting this into our utility function we see that:*

$$\begin{aligned} U\left(F, \frac{I}{p_c} - \frac{F}{p_c}p_f\right) &= \mu F + \tau\left(\frac{I}{p_c} - \frac{F}{p_c}p_f\right) \\ &= \frac{F}{p_c}\left(\mu p_c - \tau p_f\right) + \tau\frac{I}{p_c} \end{aligned}$$

and this function is linear in F*, it will be strictly increasing (and thus* $F = \frac{I}{p_f}$*) if*

$$\mu p_c > \tau p_f$$

if $\mu p_c < \tau p_f$ *it will be decreasing, and it is only indeterminate if* $\mu p_c = \tau p_f$*.*

(f) *(4 points)* Find the demand for F, you do not need to worry about the rare cases where the consumer is not at a corner solution.

Solution 19 $F = \frac{I}{p_f}$ *if* $BfB_F > BfB_C$ *or* $\frac{\mu}{p_f} > \frac{\tau}{p_c}$ *or* $\frac{1}{\tau}\mu p_c > p_f$ *thus the demand curve is:*

$$F = \begin{cases} 0 & \text{if } \frac{1}{\tau}\mu p_c < p_f \\ \textit{Indeterminate} & \text{if } \frac{1}{\tau}\mu p_c = p_f \\ \frac{I}{p_f} & \text{if } \frac{1}{\tau}\mu p_c > p_f \end{cases}$$