ECON 203
Quiz on Producer Theory

Be sure to show your work for all answers, even if the work is simple.

This exam will last 100 minutes.

1. *(6 points)* **Honor Statement:** Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person. I will also neither help others nor use a calculator or other electronic aid for calculation.

   Name and Surname: _

   Student ID: _

   Signature: _

   Remark 1 *If you write down abstract coefficients in your answer you will get no credit. If you do not show your work you will get no credit. If you give multiple answers then only the first one will be graded.*

2. *(36 points total)* In an industry there are two popular technologies, $c_A(q_A) = \frac{1}{2}q_A^2 + 9q_A + 32$ and $c_B(q_B) = 3q_B^2 + q_B + 48$. There are $n_A = 5$ firms of type A, and they all have a fixed start up costs of 18. There are $n_B = 12$ firms of type B, and they all have a fixed sunk cost of 36.

   (a) *(10 points)* Find the marginal, average variable, and average costs of firms of type A. At what price would they shut down? At what price would they enter? Finally find the firm's supply curve.

   Solution 2

$$\begin{aligned} MC_A &= q + 9 \\ AC_A &= \frac{\frac{1}{2}q^2 + 9q + 32}{q} \\ AVC_A &= \frac{\frac{1}{2}q^2 + 9q + 18}{q} \end{aligned}$$

   At P_{sd}^A marginal and average variable costs will be equal:

$$\begin{aligned} q + 9 &= \frac{\frac{1}{2}q^2 + 9q + 18}{q} \\ q_{sd}^A &= 6 \\ P_{sd}^A &= 6 + 9 = 15 \end{aligned}$$

AT P_{in}^{A} the price will be above average costs, and this will occur when average and marginal costs are equal

$$\begin{aligned} q+9 &= \frac{\frac{1}{2}q^2+9q+32}{q} \\ q_{in}^{A} &= 8 \\ P_{in}^{A} &= 8+9=17 \end{aligned}$$

Finally if the firm produces output, the firm supply curve is given by:

$$\begin{aligned} P &= MC = q+9 \\ q &= P-9\ . \end{aligned}$$

The firm will produce if $P \geq P_{sd}^{A}$. Thus the supply curve is:

$$s_A(P) = \begin{cases} P-9 & \text{if} \quad P \geq 15 \\ 0 & \text{if} \quad P \leq 15 \end{cases}$$

(b) *(10 points)* Find the marginal, average variable, and average costs of firms of type B. At what price would they shut down? At what price would they enter? Finally find the firm's supply curve.

Solution 3

$$\begin{aligned} MC_B &= 6q+1 \\ AC_B &= \frac{3q^2+q+48}{q} \\ AVC_B &= \frac{3q^2+q+48-36}{q} \end{aligned}$$

The price shut down will be the marginal cost when the margin and the average variable cost are equal:

$$\begin{aligned} 6q+1 &= \frac{3q^2+q+48-36}{q} \\ q_{sd}^{B} &= 2 \\ P_{sd}^{b} &= 6(2)+1=13 \end{aligned}$$

And the price at which these firms will enter will be when average and marginal costs are equal:

$$\begin{aligned} 6q+1 &= \frac{3q^2+q+48}{q} \\ q_{in}^{B} &= 4 \\ P_{in}^{B} &= 6(4)+1=25 \end{aligned}$$

Finally when the firm produces the supply will be given by:

$$\begin{aligned} P &= 6q+1 \\ q &= \frac{1}{6}P - \frac{1}{6} \end{aligned}$$

thus the firm level supply curve will be:

$$s_B(P) = \begin{cases} \frac{1}{6}P - \frac{1}{6} & \text{if } P \geq 13 \\ 0 & \text{if } P \leq 13 \end{cases}$$

notice the important fact that if $P = 13$ then can either produce two units or none.

(c) *(4 points)* Find the industry's short run supply curve. You may ignore any price where some of the firms do not care if they produce or not.

$$S(P) = 5s_A(P) + 12s_B(P) = \begin{cases} (5)(P-9) + (12)\left(\frac{1}{6}P - \frac{1}{6}\right) = 7P - 47 & \text{if } P > 15 \\ (12)\left(\frac{1}{6}P - \frac{1}{6}\right) = 2P - 2 & \text{if } 15 > P > 13 \\ 0 & \text{if } 13 > P \end{cases}$$

(d) *(6 points)* In the medium run, which type of firms will enter and what price will they enter at?

Solution 4 *Since $P_{in}^A = 17 < 25 = P_{in}^B$ all of the entering firms will be of type A.*

(e) *(6 points)* Write down the long run supply curve.

Solution 5

$$S(P) = \begin{cases} \infty & \text{if } P > 17 \\ n8 & \text{if } P = 17 \\ 0 & \text{if } P < 17 \end{cases}$$

where formally speaking n can be a fraction, but it is natural to think of it as an natural number ($n \in \{0, 1, 2, ...\}$)[1]

3. *(10 points total)* About cost functions:

(a) *(6 points)* What are the three necessary and sufficient conditions with regards to input prices? You may write them either mathematically or in words. If you use math you may assume there are only two inputs (L and K).

i. Non-decreasing in input prices.

ii. Homogeneous of degree one in input prices.

[1] There is some argument on whether zero is a natural number or not. I mean when was the last time you started counting something and stopped at zero?

iii. Concave in input prices.

(b) *(2 points)* What does it mean for a condition to be *necessary* in mathematics?

Solution 6 *It means that every cost function must satisfy these characteristics.*

(c) *(2 points)* What does it mean for a condition to be *sufficient* in mathematics?

Solution 7 *It means that a function that satisfies these characteristics must be a cost function.*

4. *(40 points total)* A firm has the production function $q = L^{\frac{1}{6}} K^{\frac{1}{6}}$ and want to minimize their costs where the cost of a unit of labor is $w > 0$ and the cost of a unit of capital is $r > 0$.

(a) *(4 points)* Is this production function increasing, decreasing, or constant returns to scale? Prove your answer.

Solution 8 *I will verify that it has decreasing returns to scale by showing that* $f(tL, tK) < tf(L, K)$

$$f(tL, tK) = (tL)^{\frac{1}{6}} (tK)^{\frac{1}{6}} = (t)^{\frac{1}{6}} (L)^{\frac{1}{6}} (t)^{\frac{1}{6}} (K)^{\frac{1}{6}} = (t)^{\frac{1}{3}} (L)^{\frac{1}{6}} (K)^{\frac{1}{6}} < tf(L, K) = t(L)^{\frac{1}{6}} (K)^{\frac{1}{6}}$$

if

$$\begin{aligned} (t)^{\frac{1}{3}} &< t \\ t &< t^3 \\ 1 &< t^2 \end{aligned}$$

and since $t > 1$ *this is true. Thus this function has decreasing returns to scale.*

(b) *(6 points)* Show that if $q = \sigma L^{\alpha} K^{\beta}$ then $\frac{\partial q}{\partial L} = \alpha \frac{q}{L}$ and $\frac{\partial q}{\partial K} = \beta \frac{q}{K}$. You may use this fact in the rest of the question even if you cannot prove it. **NOTE:** Since I used (α, β, σ) in the question you should in the answer.

Proof. $\frac{\partial q}{\partial L} = \alpha \sigma L^{\alpha - 1} K^{\beta} = \frac{\alpha}{L} \left(\sigma L^{\alpha} K^{\beta} \right) = \frac{\alpha}{L} q$. $\frac{\partial q}{\partial K} = \beta \sigma L^{\alpha} K^{\beta - 1} = \frac{\beta}{K} \left(\sigma L^{\alpha} K^{\beta} \right) = \frac{\beta q}{K}$ ■

(c) *(1 point)* Set up the objective function for cost minimization.

$$\mathcal{L}(L, K, \mu) = wL + rK - \mu \left(L^{\frac{1}{6}} K^{\frac{1}{6}} - q \right)$$

(d) *(3 points)* Find the first order conditions of this objective function (they will be equal to zero.)

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial L} &= w - \mu\frac{q}{6L} = 0\\ \frac{\partial \mathcal{L}}{\partial K} &= r - \mu\frac{q}{6K} = 0\\ \frac{\partial \mathcal{L}}{\partial \mu} &= -\mu L^{\frac{1}{6}}K^{\frac{1}{6}} + q = 0\end{aligned}$$

(e) *(3 points)* Find a function for labor in terms of prices and capital. It should be linear in capital.

$$\mu = 6\frac{L}{q}w = 6\frac{K}{q}r$$

$$L = K\frac{r}{w}$$

(f) *(4 points)* Find the demand for capital.

$$\begin{aligned}q &= L^{\frac{1}{6}}K^{\frac{1}{6}}\\ q &= \left(K\frac{r}{w}\right)^{\frac{1}{6}}K^{\frac{1}{6}}\\ q &= K^{\frac{1}{3}}\left(\frac{r}{w}\right)^{\frac{1}{6}}\\ q\left(\frac{w}{r}\right)^{\frac{1}{6}} &= K^{\frac{1}{3}}\\ \left(q\left(\frac{w}{r}\right)^{\frac{1}{6}}\right)^{3} &= K\\ K &= q^{3}\left(\frac{w}{r}\right)^{\frac{1}{2}}\end{aligned}$$

(g) *(3 points)* Find the demand for labor.

$$\begin{aligned}L &= K\frac{r}{w}\\ L &= q^{3}\left(\frac{r}{w}\right)^{\frac{1}{2}}\end{aligned}$$

(h) *(4 points)* Find the cost function and simplify.

$$\begin{aligned}C &= wL^{*} + rK^{*}\\ &= w\left(q^{3}\left(\frac{r}{w}\right)^{\frac{1}{2}}\right) + r\left(q^{3}\left(\frac{w}{r}\right)^{\frac{1}{2}}\right)\\ &= q^{3}\left[w^{1}r^{\frac{1}{2}}w^{-\frac{1}{2}} + r^{1}w^{\frac{1}{2}}r^{-\frac{1}{2}}\right]\\ &= q^{3}\left[w^{\frac{1}{2}}r^{\frac{1}{2}} + w^{\frac{1}{2}}r^{\frac{1}{2}}\right]\\ &= 2q^{3}w^{\frac{1}{2}}r^{\frac{1}{2}}\end{aligned}$$

(i) *(4 points)* Find marginal cost, is it increasing or decreasing? How does this relate to the returns to scale of this production function?

Solution 9

$$\begin{aligned} MC &= \frac{\partial C}{\partial q} = 3 * \left(2q^2 w^{\frac{1}{2}} r^{\frac{1}{2}}\right) \\ &= 6w^{\frac{1}{2}} r^{\frac{1}{2}} q^2 \end{aligned}$$

$$\frac{\partial MC}{\partial q} = \frac{\partial^2 C}{\partial q^2} = 12 w^{\frac{1}{2}} r^{\frac{1}{2}} q > 0$$

It is normal for a decreasing returns to scale production function to have increasing marginal cost, and since this is CES function the equivalence is precise.

(j) *(4 points)* Show that the cost function satisfies one of the three conditions with regards to input prices.

Solution 10 *The three possible characteristics you can check are:*

i. non-decreasing:

$$\begin{aligned} \frac{\partial C}{\partial w} &= \left(\frac{1}{2}\right) 2q^3 w^{\frac{1}{2}-1} r^{\frac{1}{2}} > 0 \\ \frac{\partial C}{\partial r} &= \left(\frac{1}{2}\right) 2q^3 w^{\frac{1}{2}} r^{\frac{1}{2}-1} > 0 \end{aligned}$$

ii. homogeneous of degree one:

$$\begin{aligned} c(tw, tr, q) &= 2q^3 (tw)^{\frac{1}{2}} (tr)^{\frac{1}{2}} \\ &= 2q^3 t^{\frac{1}{2}} w^{\frac{1}{2}} t^{\frac{1}{2}} r^{\frac{1}{2}} \\ &= t^{\frac{1}{2}+\frac{1}{2}} 2q^3 w^{\frac{1}{2}} r^{\frac{1}{2}} \\ &= tc(w, r, q) \end{aligned}$$

iii. concave: I bet you don't check this one. It requires the matrix of second derivatives to be negative semi-definite.

$$\begin{aligned} \frac{\partial^2 C}{\partial w^2} &= \left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right) 2q^3 w^{\frac{1}{2}-2} r^{\frac{1}{2}} \\ &= -\frac{1}{4w^2} C \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 C}{\partial r^2} &= \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) 2q^3 w^{\frac{1}{2}} r^{\frac{1}{2}-2} \\ &= -\frac{1}{4r^2} C \end{aligned}$$

these are both strictly negative, we now need to check that

$$\frac{\partial^2 C}{\partial w^2}\frac{\partial^2 C}{\partial r^2} \geq \left(\frac{\partial^2 C}{\partial r \partial w}\right)^2$$

$$\begin{aligned}
\frac{\partial^2 C}{\partial r \partial w} &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) 2q^3 w^{\frac{1}{2}-1} r^{\frac{1}{2}-1} \\
&= \frac{1}{4rw} C
\end{aligned}$$

$$\begin{aligned}
\left(-\frac{1}{4w^2}C\right)\left(-\frac{1}{4r^2}C\right) &= \frac{1}{16}\frac{C^2}{r^2 w^2} \\
\left(\frac{1}{4rw}C\right)^2 &= \frac{1}{16}\frac{C^2}{r^2 w^2}
\end{aligned}$$

so it is satisfied with equality.

(k) *(4 points)* Using the envelope theorem, find the demand for labor and verify that it agrees with the demand you found above.

$$\frac{\partial C}{\partial w} = L = \left(\frac{1}{2}\right) 2q^3 w^{\frac{1}{2}-1} r^{\frac{1}{2}} = q^3 \frac{\sqrt{r}}{\sqrt{w}}$$

which does agree with my answer above.

5. *(9 points)* In the long run we know that $P = MC = AC$, explain why.

Solution 11 *There are three conditions we must satisfy in the long run:*

(a) Profit maximization: this means that if the firm produces, then $P = MC$.

(b) No exit (shutdown): Like usual this means that $Pq - c(q) \geq -F_{su}$, *but in the long run* $F_{su} = 0$ *so this means*

$$\begin{aligned}
Pq - c(q) &\geq 0 \\
Pq &\geq c(q) \\
P &\geq \frac{c(q)}{q} = AC
\end{aligned}$$

(c) No entry: we must also be sure that we do not have a flood of new entry, thus we must have

$$\begin{aligned}
Pq - c(q) &\leq 0 \\
Pq &\leq c(q) \\
P &\leq \frac{c(q)}{q} = AC
\end{aligned}$$

Combining these conditions means that $AC \leq P \leq AC$ thus we must have $P = AC = MC$

6. *(8 points)* Prove that if $w \geq \tilde{w}$ then $c(w, r, q) \geq c(\tilde{w}, r, q)$. You may either use algebra or a graph to do the proof.

Proof. Let (L^*, K^*) be cost minimizing at (w, r) then by definition:

$$c(w, r, q) = wL^* + rK^* \geq \tilde{w}L^* + rK^*$$

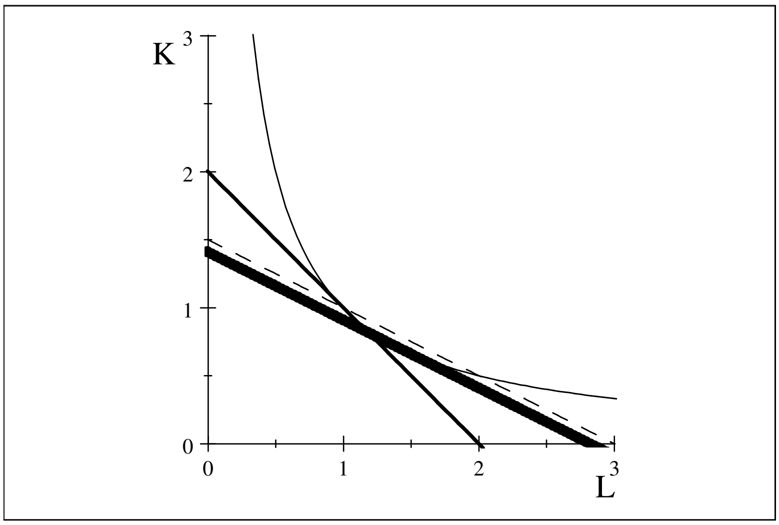
because $L^* \geq 0$ and $\tilde{w} > 0$. But, again by definition of cost mimization:

$$\tilde{w}L^* + rK^* \geq \min_{L,K} \tilde{w}L + rK = c(\tilde{w}, r, q)$$

where it is understood that the minimization is over (L, K) that can produce q units of output. ■

Proof. The (negative of the) slope of an iso-cost line is $\frac{w}{r}$, and we know that $\frac{w}{r} \geq \frac{\tilde{w}}{r}$ or the isocost curve is flatter when the price of labor falls. We also know that the vertical intercept of an isocost curve is $\frac{c}{r}$, where c is the cost.

Now let us graph an isoquant and an optimal isocost curve. I will use $q = L^{\frac{1}{2}}K^{\frac{1}{2}}$, $q = 1$, and $w = r$ then:



We can see that $\frac{c(w,r,q)}{r} = 2$, and that $L^* = K^* = 1$. When we let $\tilde{w} = \frac{1}{2}w$ and use the old input mix we can see from the graph that $\frac{\frac{1}{2}w(1)+r(1)}{r} = \frac{3}{2}$, and thus we can see that $\frac{1}{2}w(1) + r(1) < c(w, r, q)$.

By definition $c(\tilde{w}, r, q) \leq \frac{1}{2}w(1) + r(1)$, or in order to find $c(\tilde{w}, r, q)$ we must find a lower isocost curve that is above the isoquant. It seems that the approximate solution is:

$$\frac{c(\tilde{w},r,q)}{r} = 1.41,$$

but I did not work it out precisely. This proves that

$$\frac{c(w,r,q)}{r} > \frac{\frac{1}{2}w(1)+r(1)}{r} > \frac{c(\tilde{w},r,q)}{r}$$

as required. ■