Practice Questions—Chapters 9 to 11
Producer Theory
ECON 203

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These questions are to help you prepare for the exams only. Do not turn them in. Note that not all questions can be completely answered using the material in the chapter in which they are asked. These are all old exam questions and often the answers will require material from more than one chapter. Questions with lower numbers were asked in more recent years.

1 Chapter 9—Production Functions

1. Write down and define the three assumptions that we always make about the production function. For the production function $Q = \beta K^{\frac{1}{2}} + \gamma L^{\frac{1}{2}}$ show that it satisfies two of these assumptions. (You can choose which two to show.)

   (a) *Free disposal—the firm does not have to use all the inputs it has, thus* $MP_L \geq 0$ *and* $MP_K \geq 0$. $\frac{\partial Q}{\partial L} = \frac{1}{2\sqrt{L}}\gamma > 0$, $\frac{\partial Q}{\partial K} = \frac{1}{2\sqrt{K}}\beta > 0$, *so it satisifies this assumption.*

   (b) *Diminishing marginal inputs.* $\frac{\partial}{\partial L}(MP_l) = \frac{\partial^2 Q}{\partial L^2} \leq 0$. $\frac{\partial}{\partial K}(MP_k) = \frac{\partial^2 Q}{\partial K^2} \leq 0$. $\frac{\partial^2 Q}{\partial L^2} = -\frac{1}{4L^{\frac{3}{2}}}\gamma \leq 0$, $\frac{\partial^2 Q}{\partial K^2} = -\frac{1}{4K^{\frac{3}{2}}}\beta \leq 0$ *so it satisfies this assumption.*

   (c) *Convex isoquants: for* $\lambda \in [0,1]$ $f(\lambda L + (1-\lambda)L', \lambda K + (1-\lambda)K') \geq \lambda f(L,K) + (1-\lambda) f(L',K')$. *Honestly I hope no one tries to prove this, the easiest way is to look at the matrix of second derivatives. If this matrix is negative semi-definite then we are fine.*

$$\nabla Q = \begin{bmatrix} \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial L \partial K} \\ \frac{\partial^2 Q}{\partial L \partial K} & \frac{\partial^2 Q}{\partial L^2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4K^{\frac{3}{2}}}\beta & 0 \\ 0 & -\frac{1}{4L^{\frac{3}{2}}}\gamma \end{bmatrix}$$

   since $\frac{\partial^2 Q}{\partial K^2} < 0$ *and* $\frac{\partial^2 Q}{\partial L^2} < 0$ *this matrix is negative definite (note that we also need* $\frac{\partial^2 Q}{\partial K^2}\frac{\partial^2 Q}{\partial L^2} > 0$ *this is implied by* $\frac{\partial^2 Q}{\partial K^2} < 0$ *and* $\frac{\partial^2 Q}{\partial L^2} < 0$.*)* *An alternative definition is that* $\frac{\partial MRTS}{\partial K} < 0$ *where* $MRTS = \frac{MP_K}{MP_L}|_{f(L,K)=Q}$ *in order to get full credit for this you have to substitute in for the necessary quantity of labor:*

$$\frac{MP_K}{MP_L} = \frac{\frac{1}{2\sqrt{K}}\beta}{\frac{1}{2\sqrt{L}}\gamma}$$

$$\frac{MP_K}{MP_L}|_{f(L,K)=Q} = \frac{\frac{1}{2\sqrt{K}}\beta}{\frac{1}{2\sqrt{\left(\frac{Q-\beta K^{\frac{1}{2}}}{\gamma}\right)^2}}\gamma} = \frac{\frac{1}{2\sqrt{K}}\beta}{\frac{1}{2\left(\frac{Q-\beta K^{\frac{1}{2}}}{\gamma}\right)}\gamma} = \frac{1}{\sqrt{K}}\frac{\beta}{\gamma^2}\left(Q - \sqrt{K}\beta\right) = \frac{1}{\sqrt{K}}Q\frac{\beta}{\gamma^2} - \frac{\beta^2}{\gamma^2}$$

$$\frac{\partial MRTS}{\partial K} = -\frac{1}{2}\frac{1}{(K)^{\frac{3}{2}}}Q\frac{\beta}{\gamma^2} < 0$$

simply showing that

$$\frac{\partial\left(\frac{MP_K}{MP_L}\right)}{\partial K} = -\frac{1}{2}\frac{1}{(K)^{\frac{3}{2}}}\sqrt{L}\frac{\beta}{\gamma} < 0$$

is not enough.

Another simple definition is that the isoquant is bowed towards the origin, but this does not lend itself to an easy proof.

2. Consider the following production function $Q = K^{\frac{1}{2}}L^{\frac{1}{4}}$ and answer the following questions about it.

   (a) Define *free disposal.* Does this production function satisfy this assumption? Prove your answer.

   Free disposal—if a firm wishes to it can use less of its outputs than it has. I.e. if it has L units of labor it can use $0 \leq \tilde{L} \leq L$ units. This means the production function is monotonic, isoquants have negative slopes, and that the marginal products are weakly positive.

$$\frac{\partial Q}{\partial K} = \frac{1}{2}\frac{Q}{K} \geq 0$$
$$\frac{\partial Q}{\partial L} = \frac{1}{4}\frac{Q}{L} \geq 0$$

   thus this production function is monotonic.

   (b) Define *convex isoquants.* Does this production function satisfy this assumption? Argue why your answer is true. *Note: I do not require you to prove your answer for this part.*

   A *convex isoquant* is one that is bowed towards the origin, or that if $f(K,L) \geq Q$ and $f\left(\tilde{K},\tilde{L}\right) \geq Q$ and $0 \leq \alpha \leq 1$ then $f\left(\alpha K + (1-\alpha)\tilde{K}, \alpha L + (1-\alpha)\tilde{L}\right) \geq Q$, or that the MRS is decreasing as you go to the right. As to how to show this, the easiest way is to just state that this is a Cobb-Douglass function and point out we have been analyzing it all along, thus it must have convex isoquants. Another way is to show that in your graph below the isoquant is convex. Formally it will be sufficient if this production function is concave, which requires that $\frac{\partial^2 Q}{\partial K^2} = \frac{1}{2}\left(\frac{1}{2}-1\right)\frac{Q}{K^2} < 0$, $\frac{\partial^2 Q}{\partial L^2} = \frac{1}{4}\left(\frac{1}{4}-1\right)\frac{Q}{L^2} < 0$, $\frac{\partial^2 Q}{\partial K^2}\frac{\partial^2 Q}{\partial L^2} -$

$\left(\frac{\partial^2 Q}{\partial L \partial K}\right)^2 \geq 0$, where $\frac{\partial^2 Q}{\partial L \partial K} = \frac{1}{2}\frac{1}{4}\frac{Q}{LK}$. This requires that

$$\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{Q}{K^2}\frac{1}{4}\left(\frac{1}{4}-1\right)\frac{Q}{L^2} - \left(\frac{1}{8}\frac{Q}{LK}\right)^2 =$$
$$\left(\left(\frac{1}{2}-1\right)\left(\frac{1}{4}-1\right)-\frac{1}{8}\right)\frac{1}{8}\frac{Q}{K^2}\frac{Q}{L^2} \geq 0$$
$$\frac{1}{32K^2L^2}Q^2 \geq 0$$

Just work towards this answer will be enough to get credit.

(c) Define *decreasing returns to scale.* Does this production function satisfy this assumption? Prove your answer. *Note: Your answer to all other questions on this test will not be affected by your answer to this question.*

A production function satisfies *decreasing returns to scale* if when you double inputs you get less than double the output. Or more precisely for $t > 1$ $f(tK, tL) \leq tf(K, L)$. In this case this requires:

$$(tK)^{\frac{1}{2}}(tL)^{\frac{1}{4}} = t^{\frac{3}{4}}K^{\frac{1}{2}}L^{\frac{1}{4}} \leq tK^{\frac{1}{2}}L^{\frac{1}{4}}$$

or

$$t^{\frac{3}{4}} \leq t$$
$$\frac{3}{4}\ln t \leq \ln t$$
$$\frac{3}{4} \leq 1$$

3. In general when do we expect firms to have Increasing Returns to Scale? Constant Returns to Scale? Explain why we expect this in both of these cases.

   Increasing returns to scale will happen at relatively low levels of output, because at these levels of output we have to use less efficient production methods, as we increase output we switch to better production technologies. (Like changing from hand production to production line technology.)

   Constant returns to scale will happen at relatively high levels of output. At these levels of output if a firm is going to meaningfully increase output they will just build another factory, guaranteeing an equal scaling between outputs and inputs. This is called the replication hypothesis.

2 Chapter 10—Cost Functions

1. Assume that the total cost function has the arbitrary form $c(q)$, let F_{st} be the fixed startup costs and F_{su} be the fixed sunk costs. Please note

that $c(q)$ includes these costs. In this question I want you to prove that when average variable costs (AVC) are minimized average variable costs equal marginal costs (MC).

(a) In terms of this cost function write down average costs or average total costs (AC), AVC and MC.

$$AC = \frac{c(q)}{q}$$
$$AVC = \frac{c(q) - F_{su}}{q}$$
$$MC = c'(q) = \frac{\partial c(q)}{\partial q}$$

(b) What is the objective function that you need to minimize in order to minimize average variable costs?

$$\min_{q} \frac{c(q) - F_{su}}{q}$$

(c) Find the first order condition of this objective function.

$$\frac{c'(q)}{q} - \frac{c(q) - F_{su}}{q^2} = 0$$

(d) Prove that $AVC = MC$ when AVC is minimized.

$$\left(\frac{c'(q)}{q} - \frac{c(q) - F_{su}}{q^2}\right) q = 0 * q$$
$$c'(q) - \frac{c(q) - F_{su}}{q} = 0$$
$$MC - AVC = 0$$
$$MC = AVC$$

2. Assume a firm has the production function $Q = 3K^{\frac{1}{2}} + 9L^{\frac{1}{2}}$.

(a) Write down the definition of decreasing returns to scale. Does this production function satisfy decreasing returns to scale?

A production function satisfies decreasing returns to scale if for $t > 1$ $f(tL, tK) \leq tf(L, K)$.

$f(tL, tK) = 3(tK)^{\frac{1}{2}} + 9(tL)^{\frac{1}{2}} = 3t^{\frac{1}{2}}K^{\frac{1}{2}} + 9t^{\frac{1}{2}}L^{\frac{1}{2}} = t^{\frac{1}{2}}\left(3K^{\frac{1}{2}} + 9L^{\frac{1}{2}}\right) = t^{\frac{1}{2}}f(L, K)$

$$t^{\frac{1}{2}}f(L, K) < tf(L, K)$$
$$t^{\frac{1}{2}} < t$$
$$\frac{1}{2}\ln t < \ln t$$
$$\frac{1}{2} < 1$$

(b) The Long Run Cost Function.

i. Set up the objective function for the long run cost function.

$$\min_{L,K}\max_{\mu} wL + rK - \mu\left(3K^{\frac{1}{2}} + 9L^{\frac{1}{2}} - Q\right)$$

ii. Find the first order conditions.

$$w - \mu\frac{9}{2}L^{-\frac{1}{2}} = 0$$
$$r - \mu\frac{3}{2}K^{-\frac{1}{2}} = 0$$
$$-\left(3K^{\frac{1}{2}} + 9L^{\frac{1}{2}} - Q\right) = 0$$

iii. Find a function for the optimal amount of labor, L, in terms of K and input prices.

$$\mu = 2\sqrt{L}\frac{w}{9}$$
$$\mu = 2\sqrt{K}\frac{r}{3}$$

$$2\sqrt{L}\frac{w}{9} = 2\sqrt{K}\frac{r}{3}$$
$$\sqrt{L} = \frac{r9}{w3}\sqrt{K}$$
$$L = 9K\frac{r^2}{w^2}$$

iv. Find the optimal demand for capital, K.

$$3K^{\frac{1}{2}} + 9L^{\frac{1}{2}} = Q$$
$$3K^{\frac{1}{2}} + 9\left(\left(\frac{9r}{w3}\right)^2 K\right)^{\frac{1}{2}} = Q$$
$$\left(3 + 9\frac{r9}{w3}\right)K^{\frac{1}{2}} = Q$$
$$K = \frac{1}{9}Q^2\frac{w^2}{(9r+w)^2}$$

:

v. Find the optimal demand for labor, L.

$$L = \frac{r^2 9^2}{w^2 3^2}K$$
$$L = \frac{r^2 9^2}{w^2 3^2}\left(\frac{1}{9}Q^2\frac{w^2}{(9r+w)^2}\right)$$
$$= Q^2\frac{r^2}{(9r+w)^2}$$

:

vi. Find the long run cost function, and simplify if possible.

$$\begin{aligned} C^{LR} &= wL^* + rK^* \\ &= w\left(Q^2 \frac{r^2}{(9r+w)^2}\right) + r\left(\frac{1}{9}Q^2 \frac{w^2}{(9r+w)^2}\right) \\ &= Q^2 \frac{rw}{81r+9w} \end{aligned}$$

(c) The Short Run cost function. Assume throughout that the optimal demand for labor, L, is positive.

i. Set up the objective function for the short run cost function.

$$\min_L \max_\mu wL + rK - \mu\left(3K^{\frac{1}{2}} + 9L^{\frac{1}{2}} - Q\right)$$

ii. Find the first order conditions.

$$\begin{aligned} w - \mu \frac{9}{2} L^{\left(\frac{1}{2}\right)-1} &= 0 \\ -\left(3K^{\frac{1}{2}} + 9L^{\frac{1}{2}} - Q\right) &= 0 \end{aligned}$$

iii. Find the optimal amount of labor, L. Explain your methodology. *Since the second first order condition completely determines the optimal amount of labor, we only need to satisfy the production constraint to minimize our costs:*

$$\begin{aligned} 3K^{\frac{1}{2}} + 9L^{\frac{1}{2}} &= Q \\ 9L^{\frac{1}{2}} &= Q - 3K^{\frac{1}{2}} \\ L &= \frac{1}{9}K - \frac{2}{27}\sqrt{K}Q + \frac{1}{81}Q^2 \end{aligned}$$

iv. Find the short run cost function.

$$\begin{aligned} C^{SR} &= wL^* + rK \\ &= \frac{1}{81}Q^2 w + Kr + \frac{1}{9}Kw - \frac{2}{27}\sqrt{K}Qw \end{aligned}$$

v. State the general envelope theorem and explain the basic reason it is true. What does this theorem tell us that $\frac{\partial C^{sr}}{\partial K}$ is equal to? *The general envelope theorem tells us that when we take the derivative of an optimized function it is the same as the derivative of the unoptimized function that it is based on. The reason this is true that the marginal benefit and marginal cost of all indirect effects are balanced out, so the net effect of these indirect effects is zero. This tells us that* $\frac{\partial C^{sr}}{\partial K} = r - \mu MP_k$.

vi. Using the short run cost function find the optimal long run demand for capital (K), verify that it is the same as when you found the long run cost function above.

$$\frac{\partial C^{sr}}{\partial K} = \frac{1}{9^2}\left(r9^2 + w3^2 - 2\left(\frac{1}{2}\right)(K)^{\frac{1}{2}-1}Qw3\right) = 0$$

$$\frac{1}{\sqrt{K}}Qw3 = r9^2 + w3^2$$

$$Qw3 = \sqrt{K}\left(r9^2 + w3^2\right)$$

$$\frac{Qw3}{r9^2 + w3^2} = \sqrt{K}$$

$$\left(\frac{Qw3}{r9^2 + w3^2}\right)^2 = K$$

$$K = \frac{1}{9}Q^2\frac{w^2}{(9r + w)^2}$$

which is the same as above.

vii. Find the long run cost function, verify that it is the same as when you found it above.

$$C^{LR} = C^{SR}(K^*) = \frac{1}{9^2}\left(Q^2w + K^*r9^2 + K^*w3^2 - 2\sqrt{K^*}Qw3\right)$$

$$= \frac{1}{9^2}\left(Q^2w + \left(Q^2w^2\frac{3^2}{(w3^2 + r9^2)^2}\right)r9^2 + \left(Q^2w^2\frac{3^2}{(w3^2 + r9^2)^2}\right)w3^2 - 2\left(\sqrt{\left(Q^2w\right.}\right.\right.$$

$$= \frac{1}{9^2}\left(Q^2\frac{w}{w3^2 + r9^2}\left(2w3^2 + r9^2\right) - 2Qw3Qw\frac{3}{(w3^2 + r9^2)}\right)$$

$$= Q^2r\frac{w}{81r + 9w}$$

: *which is the same.*

3. For the general envelope theorem:

(a) Explain what the general envelope theorem tells us.

When taking the derivatives of an optimized function they will be the same as the derivatives of the unoptomized function upon which it is based. All indirect effects dissapear.

(b) Consider the production function $f(L, K, M)$ where M is materials, K is capital, and L is labor. Let the price of a unit of labor be w, the price of a unit of capital be r and the price of a unit of materials be μ.

i. Set up the objective function we would use to find the short run cost function, C^{sr}.

$$C^{sr} = \min_{M,L}\max_{\lambda} wL + rK + \mu M - \lambda\left(f(L, K, M) - Q\right)$$

ii. Prove the envelope theorem by finding $\frac{\partial C^{sr}}{\partial K}$.

$$\begin{aligned}\frac{\partial C^{sr}}{\partial K} &= r - \lambda f_k + w\frac{\partial L}{\partial K} + \mu\frac{\partial M}{\partial K} - \frac{\partial \lambda}{\partial K}\left(f\left(L,K,M\right) - Q\right) - \lambda f_L\frac{\partial L}{\partial K} - \lambda f_M\frac{\partial M}{\partial K} \\ &= r - \lambda f_k + \left(w - \lambda f_L\right)\frac{\partial L}{\partial K} + \left(\mu - \lambda f_M\right)\frac{\partial M}{\partial K} - \frac{\partial \lambda}{\partial K}\left(f\left(L,K,M\right) - Q\right)\end{aligned}$$

The first order conditions of the problem above are:

$$w - \lambda f_L = \mu - \lambda f_M = f\left(L,K,M\right) - Q = 0$$

thus the last three terms drop out and $\frac{\partial C^{sr}}{\partial K} = r - \lambda f_k$.

iii. How can we use $\frac{\partial C^{sr}}{\partial K}$ to find the long run cost function?
If $\frac{\partial C^{sr}}{\partial K} = r - \lambda f_k = 0$ *then we have the final first order condition for the general cost minimization problem, and if we set* K *so that this is true then we will have the long run cost function.*

4. Assume that a firm has the production function $Q = K^{\frac{2}{3}}L^{\frac{1}{6}}$. (Let the price of a unit of K be r and the price of a unit of L be w).

   (a) Define *decreasing returns to scale*, does this production function satisfy decreasing returns to scale?
   A production function satisfies decreasing returns to scale if for $\lambda > 1$ $f\left(\lambda L, \lambda K\right) \leq \lambda f\left(L,K\right)$. *For this objective function*

$$f\left(\lambda L, \lambda K\right) = \left(\lambda K\right)^{\frac{2}{3}}\left(\lambda L\right)^{\frac{1}{6}} = \lambda^{\frac{2}{3}+\frac{1}{6}}K^{\frac{2}{3}}L^{\frac{1}{6}} = \lambda^{\frac{2}{3}+\frac{1}{6}}f\left(L,K\right)$$

$$\begin{aligned}\lambda^{\frac{2}{3}+\frac{1}{6}}f\left(L,K\right) &\leq \lambda f\left(L,K\right) \\ \lambda^{\frac{2}{3}+\frac{1}{6}} &\leq \lambda \\ \frac{2}{3}+\frac{1}{6} = \frac{5}{6} &\leq 1\end{aligned}$$

   Yes

   (b) Set up the objective function to find the short run cost function, C^{SR}.

$$\min_{L}\max_{\lambda} wL + rK - \left(K^{\frac{2}{3}}L^{\frac{1}{6}} - Q\right)$$

(c) Find the short run cost function.

$$\begin{aligned}
K^{\frac{2}{3}} L^{\frac{1}{6}} &= Q \\
L^{\frac{1}{6}} &= \frac{Q}{K^{\frac{2}{3}}} \\
L &= \frac{Q^{\frac{1}{\frac{1}{6}}}}{K^{\frac{\frac{2}{3}}{\frac{1}{6}}}} \\
C^{sr} &= w\frac{Q^{\frac{1}{\frac{1}{6}}}}{K^{\frac{\frac{2}{3}}{\frac{1}{6}}}} + rK
\end{aligned}$$

$C^{sr} = Kr + \frac{1}{K^4} Q^6 w$

(d) Show that this cost function is non-increasing in input prices and homegeneous of degree one in input prices.

$$\begin{aligned}
\frac{\partial C^{sr}}{\partial w} &= \frac{Q^{\frac{1}{\frac{1}{6}}}}{K^{\frac{\frac{2}{3}}{\frac{1}{6}}}} \\
\frac{\partial C^{sr}}{\partial r} &= K
\end{aligned}$$

so it is non-decreasing in input prices.

$$\begin{aligned}
C^{sr}\left(tw, tr, K, Q\right) &= tw\frac{Q^{\frac{1}{\frac{1}{6}}}}{K^{\frac{\frac{2}{3}}{\frac{1}{6}}}} + trK \\
&= t\left(w\frac{Q^{\frac{1}{\frac{1}{6}}}}{K^{\frac{\frac{2}{3}}{\frac{1}{6}}}} + rK\right) \\
&= tC^{sr}\left(w, r, K, Q\right)
\end{aligned}$$

so it is homgeneous of degree one in input prices.

(e) Find the long run demand for Capital.

$$\frac{\partial C^{sr}}{\partial K} = -\frac{\frac{2}{3}}{\frac{1}{6}} w \frac{Q^{\frac{1}{\frac{1}{6}}}}{K^{\frac{\frac{2}{3}}{\frac{1}{6}}+1}} + r = 0$$

$$r = \frac{\frac{2}{3}}{\frac{1}{6}} w \frac{Q^{\frac{1}{\frac{1}{6}}}}{K^{\frac{\frac{2}{3}}{\frac{1}{6}}+1}}$$

$$K^{\frac{\frac{2}{3}+\frac{1}{6}}{\frac{1}{6}}} = \frac{\frac{2}{3}}{\frac{1}{6}} \frac{w}{r} Q^{\frac{1}{\frac{1}{6}}}$$

$$K = \left(\frac{\frac{2}{3}}{\frac{1}{6}} \frac{w}{r}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}}$$

(f) Find the long run cost function and show that it can be simplified to the form $C^{lr} = w^{\delta} r^{\tau} Q^{\gamma} A$ where δ, τ, γ and A are numbers that depend on the coefficients of the production function. *Note: Credit will only be given for finding the proper values of* δ, τ, γ and A.

$$C^{lr} = w \frac{Q^{\frac{1}{\frac{1}{6}}}}{\left(\left(\frac{\frac{2}{3}}{\frac{1}{6}} \frac{w}{r}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}}\right)^{\frac{\frac{2}{3}}{\frac{1}{6}}}} + r \left(\frac{\frac{2}{3}}{\frac{1}{6}} \frac{w}{r}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}}$$

$$= w \frac{Q^{\frac{1}{\frac{1}{6}}}}{\left(\frac{\frac{2}{3}}{\frac{1}{6}} \frac{w}{r}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}} \frac{\frac{2}{3}}{\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}} \frac{\frac{2}{3}}{\frac{1}{6}}}} + r \left(\frac{2/3}{\frac{1}{6}} \frac{w}{r}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}}$$

$$= w \left(\frac{\frac{1}{6}}{\frac{2}{3}} \frac{r}{w}\right)^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{1}{6}} - \frac{1}{\frac{2}{3}+\frac{1}{6}} \frac{\frac{2}{3}}{\frac{1}{6}}} + r \left(\frac{\frac{2}{3}}{\frac{1}{6}} \frac{w}{r}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}}$$

$$= w^1 w^{-\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} r^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} \left(\frac{\frac{1}{6}}{\frac{2}{3}}\right)^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}} + r^1 r^{-\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} w^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} \left(\frac{\frac{2}{3}}{\frac{1}{6}}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}}$$

$$= w^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} r^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}} \left(\frac{\frac{1}{6}}{\frac{2}{3}}\right)^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} + r^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} w^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}} \left(\frac{\frac{2}{3}}{\frac{1}{6}}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}}$$

$$= w^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} r^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}} \left[\left(\frac{\frac{1}{6}}{\frac{2}{3}}\right)^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} + \left(\frac{\frac{2}{3}}{\frac{1}{6}}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}}\right]$$

(g) Show that the long run cost function is non-decreasing in input prices and homgeneous of degree one in input prices. *Partial credit will be* *given for proving this using the general form I gave you in the last part of this problem.*

$$\frac{\partial C^{lr}}{\partial w} = \frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}} \frac{C^{lr}}{w} > 0$$
$$\frac{\partial C^{lr}}{\partial r} = \frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}} \frac{C^{lr}}{r} > 0$$

so it is non-decreasing in input prices.

$$C^{lr}(tw, tr, Q) = (tw)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} (tr)^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}} \left[\left(\frac{\frac{1}{6}}{\frac{2}{3}}\right)^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} + \left(\frac{\frac{2}{3}}{\frac{1}{6}}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} \right]$$
$$= t^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}+\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} w^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} r^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} Q^{\frac{1}{\frac{2}{3}+\frac{1}{6}}} \left[\left(\frac{\frac{1}{6}}{\frac{2}{3}}\right)^{\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} + \left(\frac{\frac{2}{3}}{\frac{1}{6}}\right)^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}} \right]$$
$$= t^{\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}}+\frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{6}}} C^{lr}(w, r, Q)$$
$$= tC^{lr}(w, r, Q)$$

(h) Find the long run demand for labor. *Partial credit will be given for finding this using the general form for the cost function I gave you above.*

$$\frac{\partial C^{lr}}{\partial w} = \frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}} \frac{C^{lr}}{w}$$

and the envelope theorem tells us that this is equal to L, thus this is the long run demand for labor. It can be simplified, but that is not necessary to get full credit for the question.

5. In this question I want you to prove that for a specific production function and in general that the cost function is homogeneous of degree one in input prices, or $C(tw, tr, Q) = tC(w, r, Q)$ for a firm that has two inputs, K or capital and L or labor.

   (a) For the production function $Q = K^{\frac{2}{3}} L^{\frac{1}{6}}$.

      i. In the graph below graph an isoquant when $Q = 1$. Be sure to list at least three points on this isoquant below.

ii. In the same graph find the cost minimizing isocost curve when $w = 1$ and $r = \frac{\frac{2}{3}}{\frac{1}{6}}$.

iii. Now in the same graph find the cost minimizing isocost curve when $w = 3$ and $r = 3\frac{\frac{2}{3}}{\frac{1}{6}}$. *It is the same.*

iv. Show in the graph that this means $C\left(3, 3\frac{\frac{2}{3}}{\frac{1}{6}}, 1\right) = 3C\left(1, \frac{\frac{2}{3}}{\frac{1}{6}}, 1\right)$. Explain how you could generalize this result to show that $C(tw, tr, Q) = tC(w, r, Q)$.

On the graph we can see that

$$\frac{C\left(1, \frac{\frac{2}{3}}{\frac{1}{6}}, 1\right)}{\frac{\frac{2}{3}}{\frac{1}{6}}} = \frac{C\left(3, 3\frac{\frac{2}{3}}{\frac{1}{6}}, 1\right)}{3\frac{\frac{2}{3}}{\frac{1}{6}}}$$

$$3C\left(1, \frac{\frac{2}{3}}{\frac{1}{6}}, 1\right) = C\left(3, 3\frac{\frac{2}{3}}{\frac{1}{6}}, 1\right)$$

as required. Obviously the general insight does not depend on the value of (w, r, Q) so the result should hold in general.

(b) Now I want you to show the same thing for general production processes. Let (K^*, L^*) be the cost minimizing bundle at the input prices (w, r) and output Q.

i. If arbitrary (K, L) produce at least Q units of output is $wL^* + rK^* \geq wL + rK$, $wL^* + rK^* \leq wL + rK$, or are we unable to tell? Explain.

$$wL^* + rK^* \leq wL + rK$$

by definition of cost minimization.

ii. Prove that the cost function is homogenous of degree one in input prices, or that for $t > 0$, $tC(w, r, Q) = C(tw, tr, Q)$.
Since $t > 0$

$$t(wL^* + rK^*) \leq t(wL + rK)$$
$$twL^* + trK^* \leq twL + trK$$

so (K^*, L^*) *are cost minimizing at* (tw, tr) *thus:*

$$C(tw, tr, Q) = twL^* + trK^* = t(wL^* + rK^*) = tC(w, r, Q)$$

6. A firm has three inputs, capital (K), labor (L) and pollution or bad air (B) their production function is $Q = f(K, L, B)$ the government regulates the firm buy saying that B must be less than some given $\bar{B}$, or mathematically that $B \leq \bar{B}$. Note that you will get severely reduced credit if you assume a functional form for $f(K, L, B)$.

   (a) What standard assumption do we need to make on the production function to insure that $B = \bar{B}$ in any cost minimizing solution?

   *If the production function satisfies **free disposal** this will be true. I.e. one can always produce less air pollution than one was planning on. This guarantees that the marginal products are non-negative.*

   (b) Set up the firm's cost minimizing objective function, be sure that you include *two* constraints. You may assume that the assumption you needed for the last part of this question is true.

$$C(w, r, Q, \bar{B}) = \min_{L,K,B} wL + rK - \lambda(f(K, L, B) - Q) - \mu(B - \bar{B})$$

   (c) Find the first order conditions of the cost minimization problem.

$$w - \lambda f_l = 0$$
$$r - \lambda f_k = 0$$
$$\lambda f_b - \mu = 0$$
$$f(K, L, B) - Q = 0$$
$$B - \bar{B} = 0$$

   (d) State the envelope theorem, or tell me what the envelope theorem implies.

   There are many ways to do this. In essence it says that the derivatives of an optimized function will be the same as the derivatives of the unoptimized function that it is derived from.

(e) Prove the envelope theorem by finding the derivative of this firm's cost function with respect to $\bar{B}$.

$$C\left(w,r,Q,\bar{B}\right) = \min_{L,K,B} wL + rK - \lambda\left(f\left(K,L,B\right) - Q\right) - \mu\left(B - \bar{B}\right)$$

$$\frac{\partial C}{\partial \bar{B}} = \mu + w\frac{\partial L}{\partial \bar{B}} + r\frac{\partial K}{\partial \bar{B}} - \frac{\partial \lambda}{\partial \bar{B}}\left(f\left(K,L,B\right) - Q\right) - \frac{\partial \mu}{\partial \bar{B}}\left(B - \bar{B}\right)$$
$$- \lambda f_l \frac{\partial L}{\partial \bar{B}} - \lambda f_k \frac{\partial K}{\partial \bar{B}}$$

from the first order conditions $\left(f\left(K,L,B\right) - Q\right) = \left(B - \bar{B}\right) = 0$, *and then we can rearrange the derivatives to be:*

$$\frac{\partial C}{\partial \bar{B}} = \mu + \left(w - \lambda f_l\right)\frac{\partial L}{\partial \bar{B}} + \left(r - \lambda f_k\right)\frac{\partial K}{\partial \bar{B}}$$

and we can see that $\left(w - \lambda f_l\right) = \left(r - \lambda f_k\right) = 0$ *thus we can see that:*

$$\frac{\partial C}{\partial \bar{B}} = \mu = \frac{\partial}{\partial \bar{B}}\left[wL + rK - \lambda\left(f\left(K,L,B\right) - Q\right) - \mu\left(B - \bar{B}\right)\right]$$

thus proving the envelope theorem.

(f) The government wants to find out the firm's marginal benefit of pollution (B) is not balanced with the marginal social cost of pollution. Assuming they do not know the firm's production function how can they find out what this marginal benefit is? What would they need to do to find this benefit?

By using the envelope theorem we can see that $\frac{\partial C}{\partial \bar{B}} = \mu$ *which is the marginal benefit of pollution to this firm. In order to find this derivative they should vary* $\bar{B}$ *somewhat, to find out what this marginal impact is.*

7. In this question I want you to prove properties of the cost function for a firm that has two inputs, K or capital and L or labor. Let (K^*, L^*) be the cost minimizing bundle at the input prices (w, r) and output Q.

(a) If arbitrary (K, L) produce at least Q units of output is $wL^* + rK^* \geq wL + rK$, $wL^* + rK^* \leq wL + rK$, or are we unable to tell? Explain.

$$wL^* + rK^* \leq wL + rK$$

because by definition the cost minimizing bundle is cheaper than every other bundle.

(b) Prove that input demands are homogenous of degree zero in input prices, or that for $t > 0$, $K\left(w,r,Q\right) = K\left(tw,tr,Q\right)$ and $L\left(w,r,Q\right) = L\left(tw,tr,Q\right)$.

This means that if $wL^* + rK^* \leq wL + rK$ *then for every* $t > 0$ $twL^* + trK^* \leq twL + trK$. *However the latter statement can be simplified:*

$$\begin{aligned} twL^* + trK^* &\leq twL + trK \\ t\left(wL^* + rK^*\right) &\leq t\left(wL + rK\right) \\ wL^* + rK^* &\leq wL + rK \end{aligned}$$

until it is the former statement. Thus the input demands are homogenous of degree zero.

(c) Prove that the cost function is homogenous of degree one in input prices, or that for $t > 0$, $tC\left(w, r, Q\right) = C\left(tw, tr, Q\right)$.

$$\begin{aligned} C\left(tw, tr, Q\right) &= twL\left(tw, tr, Q\right) + trK\left(tw, tr, Q\right) \\ &= twL\left(w, r, Q\right) + trK\left(w, r, Q\right) \\ &= t\left(wL\left(w, r, Q\right) + rK\left(w, r, Q\right)\right) \\ &= tC\left(w, r, Q\right) \end{aligned}$$

as required. Of course one could answer the last question as a way to answer this one. There are many other ways of writing the same thing, but the analysis always boils down to the same fact.

8. Given that a firm has the production function $Q = 8\sqrt{K} + 2\sqrt{L}$ find the long run cost function. (Let the price of a unit of K be r and the price of a unit of L be w).

   (a) Define *free disposal*, does this production function satisfy free disposal?

   Free disposal means that a firm can always use less of an input then it has, this implies that the marginal products are positive. In this problem

$$\begin{aligned} MP_L &= \frac{1}{\sqrt{L}} \\ MP_K &= \frac{1}{\sqrt{K}}4 \end{aligned}$$

   which are both positive so yes it does.

   (b) Set up the objective function.

$$\min_{L,K} \max_{\lambda} wL + rK - \lambda\left(2\sqrt{K}4 + 2\sqrt{L}1 - Q\right)$$

(c) Find the first order conditions.

$$\begin{aligned} w - \lambda 1 L^{\frac{1}{2}-1} &= 0 \\ r - \lambda 4 K^{\frac{1}{2}-1} &= 0 \\ 2\sqrt{K}4 + 2\sqrt{L}1 - Q &= 0 \end{aligned}$$

(d) Find the "buck for the bang" of Capital (K) and Labor (L).

$$\frac{w}{1L^{\frac{1}{2}-1}} = \lambda = \frac{r}{4K^{\frac{1}{2}-1}}$$

(e) Find a formula for Labor in terms of Capital and input prices.

$$\begin{aligned} \frac{wL^{1-\frac{1}{2}}}{1} &= \frac{rK^{1-\frac{1}{2}}}{4} \\ L^{1-\frac{1}{2}} &= \frac{r1}{4w}K^{1-\frac{1}{2}} \\ L &= \left(\frac{r1}{4w}\right)^{\frac{1}{1-\frac{1}{2}}} K \\ &= K\frac{r^2}{w^2}\frac{1^2}{4^2} \end{aligned}$$

(f) Find the demand for Capital.

$$\begin{aligned} \frac{1}{\frac{1}{2}}\left(\left(\frac{r1}{4w}\right)^{\frac{1}{1-\frac{1}{2}}} K\right)^{\frac{1}{2}} + \frac{4}{\frac{1}{2}}K^{\frac{1}{2}} - Q &= 0 \\ \left(\frac{1}{\frac{1}{2}}\left(\frac{r1}{4w}\right)^{\frac{\frac{1}{2}}{1-\frac{1}{2}}} + \frac{4}{\frac{1}{2}}\right)K^{\frac{1}{2}} &= Q \\ K &= \frac{1}{4}Q^2w^2\frac{4^2}{\left(r1^2+w4^2\right)^2} \end{aligned}$$

(g) Find the demand for Labor.

$$\begin{aligned} L &= K\frac{r^2}{w^2}\frac{1^2}{4^2} \\ L &= \frac{r^2}{w^2}\frac{1^2}{4^2}\frac{1}{4}Q^2w^2\frac{4^2}{\left(r1^2+w4^2\right)^2} \\ &= \frac{1}{4}Q^2r^2\frac{1^2}{\left(r1^2+w4^2\right)^2} \end{aligned}$$

(h) Solve for the long run cost function.

$$\begin{aligned}
C &= wL + rK \\
&= w\frac{1}{4}Q^2r^2\frac{1^2}{\left(r1^2+w4^2\right)^2} + r\frac{1}{4}Q^2w^2\frac{4^2}{\left(r1^2+w4^2\right)^2} \\
&= \frac{1}{4}Q^2\frac{rw}{r1^2+w4^2}
\end{aligned}$$

9. For the production function $Q = 8\sqrt{K} + 2\sqrt{L}$

(a) Find the firm's short run demand for labor and their short run cost function, $C^{sr}(r, w, K, Q)$. You may assume that output is high enough that labor must be used.

$$\begin{aligned}
Q &= \frac{1}{\frac{1}{2}}L^{\frac{1}{2}} + \frac{4}{\frac{1}{2}}K^{\frac{1}{2}} \\
Q - \frac{4}{\frac{1}{2}}K^{\frac{1}{2}} &= \frac{1}{\frac{1}{2}}L^{\frac{1}{2}} \\
\frac{1}{4}\left(Q - 2\sqrt{K}4\right)^2 &= L
\end{aligned}$$

$$C^{sr}(r, w, K, Q) = w\frac{1}{4}\left(Q - 2\sqrt{K}4\right)^2 + rK$$

(b) *(3 points)* Using the short run cost function and the envelope theorem find the optimal amount of capital.

$$\frac{\partial C^{SR}}{\partial K} = -\left(\frac{1}{2}\right)2w\frac{1}{4}\left(Q - 2\sqrt{K}4\right)K^{\frac{1}{2}-1}4 + r$$

$$\begin{aligned}
r - \frac{1}{\sqrt{K}}w4\frac{1}{4}\left(\frac{1}{2}Q - \sqrt{K}4\right) &= 0 \\
\sqrt{K}r &= w4\frac{1}{4}\left(Q\frac{1}{2} - \sqrt{K}4\right) \\
\sqrt{K}4\frac{r}{w}\frac{1^2}{4} &= Q\frac{1}{2} - \sqrt{K}4 \\
\sqrt{K}4\frac{r}{w}\frac{1^2}{4} + \sqrt{K}4 &= Q\frac{1}{2} \\
K^{\frac{1}{2}} &= \frac{Q\frac{1}{2}}{4\frac{r}{w}\frac{1^2}{4}+4} \\
K &= \frac{1}{4}Q^2w^2\frac{4^2}{\left(4r1^2+w4^2\right)^2}
\end{aligned}$$

(c) *(1 point)* Verify that the optimal amount of capital you just found is the same as the long run demand for capital you found in question *2*.

The point of this question is merely to be sure that you check. Working with the short run costs we get

$$K = \frac{1}{4}Q^2 w^2 \frac{4^2}{\left(4r1^2 + w4^2\right)^2}$$

working with long run costs we get:

$$K = \frac{1}{4}Q^2 w^2 \frac{4^2}{\left(r1^2 + w4^2\right)^2}$$

and these are clearly the same.

10. I claim that in general the total cost function contains both economic costs and accounting costs. Define *economic costs* and *accounting costs* and explain.

    An **economic cost** *is a cost like opportunity cost that has to do with the future possible uses of a good. Indeed one could say that an opportunity cost is economic costs.*

    An **accounting cost** *is a cost that has already been paid. Examples are sunk costs, but it would also include any wage bills for workers, etceteras.*

    The total cost function always includes sunk costs, which are clearly accounting costs and not economic costs, so it often includes accounting costs. It is also based on the opportunity costs of not using resources as they would be in production, especially for resources owned by the owner of the firm, these opportunity costs are economic costs, so the statement is true.

11. About Economists and Accountants. **Define all technical terms you use and explain all of your answers.**

    (a) What is the fundamental difference between economists and accountants?

    Of course a question like this will lead to a large variety of answers, and I will have to accept many different types of answers. One could say it is the task of economists and accountants. Accountants are supposed to tell investors and the government what happened, Economists are supposed to offer advice on what to do next. This means that Accountants care about history and Economists only care about the future.

    (b) Which type of costs do accountants think are important but economists do not? Which type of costs do economists think are important but economists do not?

Accountants care about sunk costs, non-recoverable costs. These are events that have occurred and can not be changed, exactly the type of information investors need to know. Economists care about opportunity costs, or the market value of any good or service a firm uses. This is what the firm owner could get by not using the good or service, and is the appropriate guide to decision making.

(c) What does this tell us about the relationship between accounting profit and economic profit in the long run?

Economic profit will be zero because there are no sunk costs in the long run. Thus every input used must be getting its opportunity cost. Accounting profit will be positive, because it does not include inputs that are owned by the owner of the firm. Thus Economic Profit is lower than Accounting profit.

(d) What relationship would we expect between accounting profit and economic profit in the short run? (Your answer can be that you can not tell, but you must explain why each case could be true.)

The answer is that it could be either. Economic profit will be higher when the sunk costs are lower than the opportunity costs of the owner's labor and capital. Accounting profit will be higher if the reverse is true. Thus in the absolute short run (when all costs are sunk) clearly Economic profit will be higher than Accounting profit, and as the time frame lengthens relatively speaking Economic profit will decline.

12. Given that a firm has the production function $Q = K^{\frac{1}{2}} L^{\frac{1}{4}}$ find the long run cost function. (Let the price of a unit of K be r and the price of a unit of L be w).

(a) Set up the objective function.

$$\min_{L,K} \max_{\mu} wL + rK - \mu \left(K^{\frac{1}{2}} L^{\frac{1}{4}} - Q \right)$$

(b) Find the first order conditions.

$$w - \mu \frac{1}{4} \frac{Q}{L} = 0$$
$$r - \mu \frac{1}{2} \frac{Q}{K} = 0$$
$$-\left(K^{\frac{1}{2}} L^{\frac{1}{4}} - Q \right) = 0$$

(c) Find the "buck for the bang" of Capital (K) and Labor (L).

$$\mu = \frac{L}{Q} \frac{w}{\frac{1}{4}}$$
$$\mu = \frac{K}{Q} \frac{r}{\frac{1}{2}}$$

(d) Using the first order conditions find a formula for Labor in terms of Capital and input prices.

$$\frac{L}{Q}\frac{w}{\frac{1}{4}} = \frac{K}{Q}\frac{r}{\frac{1}{2}}$$
$$L = K\frac{r}{w2}$$

(e) Find the demand for Capital.

$$-\left(K^{\frac{1}{2}}L^{\frac{1}{4}} - Q\right) = 0$$
$$-\left(K^{\frac{1}{2}}\left(K\frac{r}{2w}\right)^{\frac{1}{4}} - Q\right) = 0$$
$$Q = K^{\frac{1}{2}+\frac{1}{4}}\left(\frac{r}{2w}\right)^{\frac{1}{4}}$$
$$Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{2w}{r}\right)^{\frac{1}{2+1}} = K$$

(f) Find the demand for Labor.

$$L = K\frac{r}{2w} = Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{2w}{r}\right)^{\frac{1}{2+1}}\frac{r}{2w}$$
$$= Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{r}{2w}\right)^{1-\frac{1}{2+1}} = Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{r}{2w}\right)^{\frac{2}{2+1}}$$

(g) Solve for the long run cost function.

$$C = wL + rK$$
$$= wQ^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{r}{2w}\right)^{\frac{2}{2+1}} + rQ^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{2w}{r}\right)^{\frac{1}{2+1}}$$
$$= w^{1-\frac{2}{2+1}}Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}r^{\frac{2}{2+1}}\left(\frac{1}{2}\right)^{\frac{2}{2+1}} + r^{1-\frac{1}{2+1}}Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}w^{\frac{1}{2+1}}(2)^{\frac{1}{2+1}}$$
$$= Q^{\frac{1}{\frac{1}{4}(2+1)}}r^{\frac{2}{2+1}}w^{\frac{1}{2+1}}\left(2^{\frac{1}{2+1}} + \left(\frac{1}{2}\right)^{\frac{2}{2+1}}\right)$$

or,

$$
\begin{aligned}
C &= wL + rK \\
L &= K\frac{r}{2w} \\
C &= w\left(K\frac{r}{2w}\right) + rK = rK\left(1+\frac{1}{2}\right) \\
K &= Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{2w}{r}\right)^{\frac{1}{2+1}} \\
C &= rQ^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{2w}{r}\right)^{\frac{1}{2+1}}\left(1+\frac{1}{2}\right) \\
&= r^{1-\frac{1}{2+1}}w^{\frac{1}{2+1}}Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}2^{\frac{1}{1+2}}\left(1+\frac{1}{2}\right) \\
&= \frac{3}{2}\sqrt[3]{2}Q^{\frac{4}{3}}r^{\frac{2}{3}}\sqrt[3]{w}
\end{aligned}
$$

(h) Show that the cost function is homogenous of degree one in input prices and increasing in input prices.

$$
\begin{aligned}
C\left(tw, tr, Q\right) &= Q^{\frac{1}{\frac{1}{4}(2+1)}}\left(tr\right)^{\frac{2}{2+1}}\left(tw\right)^{\frac{1}{2+1}}\left(2^{\frac{1}{2+1}} + \left(\frac{1}{2}\right)^{\frac{2}{2+1}}\right) \\
&= t^{\frac{2}{2+1}+\frac{1}{2+1}}Q^{\frac{1}{\frac{1}{4}(2+1)}}r^{\frac{2}{2+1}}w^{\frac{1}{2+1}}\left(2^{\frac{1}{2+1}} + \left(\frac{1}{2}\right)^{\frac{2}{2+1}}\right) \\
&= tQ^{\frac{1}{\frac{1}{4}(2+1)}}r^{\frac{2}{2+1}}w^{\frac{1}{2+1}}\left(2^{\frac{1}{2+1}} + \left(\frac{1}{2}\right)^{\frac{2}{2+1}}\right) \\
&= tC\left(w, r, Q\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C}{\partial w} &= \frac{1}{2+1}Q^{\frac{1}{\frac{1}{4}(2+1)}}r^{\frac{2}{2+1}}w^{\frac{1}{2+1}-1}\left(2^{\frac{1}{2+1}} + \left(\frac{1}{2}\right)^{\frac{2}{2+1}}\right) \geq 0 \\
\frac{\partial C}{\partial r} &= \frac{2}{2+1}Q^{\frac{1}{\frac{1}{4}(2+1)}}r^{\frac{2}{2+1}-1}w^{\frac{1}{2+1}}\left(2^{\frac{1}{2+1}} + \left(\frac{1}{2}\right)^{\frac{2}{2+1}}\right) \geq 0
\end{aligned}
$$

13. For the production function $Q = K^{\frac{1}{2}}L^{\frac{1}{4}}$

(a) Set up the objective function for the short run cost function.

$$
\min_{L}\max_{\mu} wL + rK - \mu\left(K^{\frac{1}{2}}L^{\frac{1}{4}} - Q\right)
$$

(b) Find the firm's short run demand for labor and their short run cost function, $C^{sr}(r, w, K, Q)$.

$$K^{\frac{1}{2}}L^{\frac{1}{4}} = Q$$

$$L = \left(\frac{Q}{K^{\frac{1}{2}}}\right)^{\frac{1}{\frac{1}{4}}} = \frac{Q^{\frac{1}{\frac{1}{4}}}}{K^2}$$

$$C^{sr} = wL + rK = w\frac{Q^{\frac{1}{\frac{1}{4}}}}{K^2} + rK$$

(c) Using the short run cost function and the envelope theorem find the optimal amount of capital.

$$\frac{\partial C^{sr}}{\partial K} = -2w\frac{Q^{\frac{1}{\frac{1}{4}}}}{K^{2+1}} + r = 0$$

$$r = 2w\frac{Q^{\frac{1}{\frac{1}{4}}}}{K^{2+1}}$$

$$K = \left(\frac{2w}{r}Q^{\frac{1}{\frac{1}{4}}}\right)^{\frac{1}{2+1}}$$

(d) Verify that the optimal amount of capital you just found is the same as the long run demand for capital you found in the last question.

This question really is just to give you an incentive to do the check. Of course they are the same in my answers.

14. In a graph:

(a) Draw and label the isoquant where $Q = 8$ for the production function $Q = K^{\frac{1}{2}}L^{\frac{1}{4}}$.

In general the formula for the isoquant is:

$$Q^{\frac{1}{\frac{1}{4}}} = K^{2\frac{\frac{1}{4}}{\frac{1}{4}}}L$$

$$L = \frac{Q^{\frac{1}{\frac{1}{4}}}}{K^2}$$

(b) If $w = 3$ and $r = 6$ draw and label the isocost line that is just tangent to the isoquant you found. Be sure to label the points where the isocost curve crosses the axes and the optimal amount of Labor and Capital.

$w = 3$ and $r_o = 6$, and I can get the demands and cost function from the question above:

$$L = Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{r}{w2}\right)^{\frac{2}{2+1}} = \frac{1}{2}\sqrt[3]{2}Q^{\frac{4}{3}}\left(\frac{r}{w}\right)^{\frac{2}{3}} = \frac{1}{2}\sqrt[3]{2}Q^{\frac{4}{3}}\left(\frac{6}{3}\right)^{\frac{2}{3}} = Q^{\frac{4}{3}} = (8)^{\frac{4}{3}} = 16$$

$$K = Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{w2}{r}\right)^{\frac{1}{2+1}} = Q^{\frac{1}{\frac{1}{4}}\frac{1}{2+1}}\left(\frac{(3)\,2}{6}\right)^{\frac{1}{2+1}} = Q^{\frac{4}{3}} = 16$$

$$C = 3L + 6K = 3\,(16) + 6\,(16) = 144$$

$$144 = 3L + 6K$$

of course I don't expect your graph to be that exact, but it could be. The Isocost line is the slanted line in the graph, the optimal inputs are indicated by the heavy straight lines.

(c) In the same graph draw the cost minimizing isocost line when w decreases to $\frac{3}{2^{\frac{3}{2}}}\tilde{\ }1.06$ (and $r = 6$).

(d) Using this graph prove that if $w \geq \tilde{w}$ then $C\,(w, r, Q) \geq C\,(\tilde{w}, r, Q)$, or that the cost function is non-decreasing in input prices.

What you need to do is draw an isocost curve which cuts through the old bundle at the new prices. This would have a cost of $\frac{9'}{6}$, *graphically it will cut somewhere between the old cost,* $\frac{C^0}{r}$ *and the new cost,* $\frac{\tilde{C}}{r}$. *The proof is that clearly* $\frac{C'}{r} < \frac{C^0}{r}$, *and then* $\frac{\tilde{C}}{r} \leq \frac{C'}{r}$ *by cost minimization. Many of you wrote something wrong for this part and then answered the next part of the question by proving the statement algebraically. I have given you 6 points for that (as if you had answered this part correctly) since that gives you a higher total.*

(e) Explain how you could generalize this graph to show that if $w \geq \tilde{w}$ and $r \geq \tilde{r}$ then $C(w, r, Q) \geq C(\tilde{w}, \tilde{r}, Q)$.

We have already shown that if $w \geq \tilde{w}$ *then* $C(w, r, Q) \geq C(\tilde{w}, r, Q)$ *now we need to draw a second graph where* $r \geq \tilde{r}$ *then we can show that* $C(\tilde{w}, r, Q) \geq C(\tilde{w}, \tilde{r}, Q)$ *combining the two graphs shows the general condition,* $C(w, r, Q) \geq C(\tilde{w}, r, Q) \geq C(\tilde{w}, \tilde{r}, Q)$

15. Assume that a firm has a production function $Q = f(K, L, M, E)$ where K is capital with a price of r, L is labor with a price of w, M is materials with a price of p_m and E is energy with a price of p_e.

(a) State the envelope theorem, or tell me what the envelope theorem implies.

There are many wordings of this, but the simplest is that when taking the derivative of an optimized function it is the same as if you took the derivative of the unoptimized function. In other words the indirect effect (the effect on the chosen variables) will disappear.

(b) Set up this firms' cost minimization problem.

$$C(r, w, p_m, p_e, Q) = \max_{\lambda} \min_{K,L,M,E} rK + wL + p_m M + p_e E + \lambda (Q - f(K, L, M, E))$$

(c) Prove the envelope theorem by finding the derivative of this firm's cost function with respect to p_m. You will not get credit if you assume that the production function has a specific functional form.

The first order conditions are:

$$\begin{aligned} r - \lambda f_K &= 0 \\ w - \lambda f_L &= 0 \\ p_m - \lambda f_M &= 0 \\ p_e - \lambda f_E &= 0 \\ Q - f(K, L, M, E) &= 0 \end{aligned}$$

When I take the derivative of the function above:

$$
\begin{aligned}
\frac{\partial C}{\partial p_e} &= M + r\frac{\partial K}{\partial p_e} + w\frac{\partial L}{\partial p_e} + p_m\frac{\partial M}{\partial p_e} + p_e\frac{\partial E}{\partial p_e} + \frac{\partial \lambda}{\partial p_e}\left(Q - f\left(K, L, M, E\right)\right) \\
&- \lambda f_K\frac{\partial K}{\partial p_e} - \lambda f_L\frac{\partial L}{\partial p_e} - \lambda f_M\frac{\partial M}{\partial p_e} - \lambda f_E\frac{\partial E}{\partial p_e} \\
&= M + \frac{\partial K}{\partial p_e}\left(r - \lambda f_K\right) + \frac{\partial L}{\partial p_e}\left(w - \lambda f_L\right) + \frac{\partial M}{\partial p_e}\left(p_m - \lambda f_M\right) + \frac{\partial E}{\partial p_e}\left(p_e - \lambda f_E\right) \\
&+ \frac{\partial \lambda}{\partial p_e}\left(Q - f\left(K, L, M, E\right)\right)
\end{aligned}
$$

and since by the first order conditions $(r - \lambda f_K) = (w - \lambda f_L) = (p_m - \lambda f_M) = (p_e - \lambda f_E) = (Q - f(K, L, M, E)) = 0$ *this simplifies to* $\frac{\partial C}{\partial p_e} = M$, *which is the derivative of the unoptimized function.*

16. About the envelope theorem.

 (a) Explain what the envelop theorem tells us simply enough that your 10 year old sibling could understand. (2 points will only be awarded if you do not use the term "derivative" in your answer. You may use the terms "function" and "optimized" or "optimal.")

 For an optimized function the impact of a change is the same as if the function was not optimized. Not really good enough for a 10 year old, but fairly simple.

 Best Student Answer, and better than mine: Say that the price of candy bars increases, how much should your allowance increase? If your parents want to keep you just as happy as before it should be just enough so that you could buy everything you were before. I.e. $\Delta p_{candybars} * \#Candybars$.

 (b) Explain why the statement you made above is true, you may either use math or English.

 In words it is because all of the indirect effects will be wiped out, the marginal cost and benefit of those effects will be balanced out meaning that the only effect will be the direct effect.

 Mathematically let:

$$V\left(w, r, p_x, p_y\right) = \min_{L,K} \max_{X,Y} F\left(L, K, X, Y, w, r, p_x, p_y\right)$$

 then the first order conditions of the problem are $F_L = F_K = F_X = F_Y = 0$ *thus:*

$$
\begin{aligned}
\frac{\partial V}{\partial w} &= \frac{\partial F}{\partial w} + F_L\frac{\partial L}{\partial w} + F_K\frac{\partial K}{\partial w} + F_X\frac{\partial X}{\partial w} + F_Y\frac{\partial Y}{\partial w} \\
&= \frac{\partial F}{\partial w}
\end{aligned}
$$

(c) If $C^{SR}(w, r, K, Q)$ is a short run cost function what does this theorem tell us $\frac{\partial C^{SR}}{\partial K}$ is equal to? How can we use this to find the long run cost function $C(w, r, Q)$?

By this theorem, $\frac{\partial C^{SR}}{\partial K} = r - \lambda MP_K$*, and thus if* $\frac{\partial C^{SR}}{\partial K} = 0$ *this is the optimal level of capital for this* $\{w, r, Q\}$ *and this tells us what* $C(w, r, Q)$ *is.*

17. This question is about proving the fact that if $w_n \geq w_o$ and $r_n \geq r_o$ then $C(w_n, r_n, Q) \geq C(w_o, r_o, Q)$.

(a) In the graph below the straight line is the isocost curve given they need to produce the level of output represented by the isoquant. Find and label the optimal level of labor and capital L_o and K_o respectively, also label the point where the isocost curve crosses the L axis in terms of the general costs, C_o and the price of a unit of labor, w_o.

(b) In the same graph draw a new isocost curve where the only difference is that the price of capital (r) increases. Prove that the new cost (C_n) must be higher than the old cost (C_o) using this graph.

This is the thick isocost curve in the graph above. The point where it crosses the vertical axis is $\frac{C_n}{w_o}$ *and from the graph it is clear that* $\frac{C_n}{w_o} > \frac{C_o}{w_o}$ *so* $C_n > C_o$*.*

(c) Explain why the above graph also proves that if both r and w increase then we must have $C(w_n, r_n, Q) \geq C(w_o, r_o, Q)$.

We can look at each price change in isolation, so the graph above shows us that $C(w_o, r_n, Q) \geq C(w_o, r_o, Q)$ *and that* $C(w_n, r_n, Q) \geq C(w_o, r_n, Q)$*, we can combine them to get the general statement.*

(d) Does the isoquant in the graph on the last page satisfy free disposal? Convexity? Explain your answer. *(Note that the fact that isoquants in general may not satisfy these properties that makes the formal proof below necessary.)*

Free disposal—the isoquant is downward sloping so it satisfies free disposal.

Convexity—drawing a line between any two points on the isoquant results in a line that is always above the isoquant, so it satisfies convexity.

(e) Let $\{L_n, K_n\}$ be the cost minimizing inputs at $\{w_n, r_n, Q\}$. For other $\{L, K\}$ that will produce at least Q units of output is it true that $w_n L_n + r_n K_n \leq w_n L + r_n K$, $w_n L_n + r_n K_n \geq w_n L + r_n K$ or are you unable to tell? Why is the inequality the way that it is?

By the definition of cost minimization $w_n L_n + r_n K_n \leq w_n L + r_n K$*.*

(f) Explain why $w_n L_n + r_n K_n \geq w_o L_n + r_o K_n$. *(This is an easy question, answering it does not require understanding or answering the rest of the question.)*

Since $w_n \geq w_o$ and $r_n \geq r_o$ *we have* $w_n L_n + r_n K_n \geq w_o L_n + r_o K_n$.

(g) Let $\{L_o, K_o\}$ be the cost minimizing inputs at $\{w_o, r_o, Q\}$. Is $w_o L_n + r_o K_n \leq w_o L_o + r_o K_o$, $w_o L_n + r_o K_n \geq w_o L_o + r_o K_o$ or are you unable to tell? Why is the inequality the way that it is?

Just like before $w_o L_n + r_o K_n \geq w_o L_o + r_o K_o$ *by the definition of cost minimization.*

(h) Combining the statements you just proved; prove the general statement, that if $w_n \geq w_o$ and $r_n \geq r_o$ then $C(w_n, r_n, Q) \geq C(w_o, r_o, Q)$.

$$C(w_n, r_n, Q) = w_n L_n + r_n K_n \geq w_o L_n + r_o K_n \geq w_o L_o + r_o K_o = C(w_o, r_o, Q)$$

(i) In this proof why did I start with $\{L_n, K_n\}$ instead of $\{L_o, K_o\}$?

In some ways this is most important part of the question. We start with $\{L_n, K_n\}$ *because since we are minimizing it is always easier to go down. If we started with* $\{L_o, K_o\}$ *then minimization would have been working in the opposite direction of the way we wanted the inequalities to go.*

18. In the long run accounting costs are lower than economic costs, in the short run economic costs are often lower than accounting costs. Explain why this is true.

Accounting costs do not include opportunity costs, thus in the long run when there are no sunk costs economic costs must be higher. However accounting costs do include sunk costs, thus often in the short run they are higher than economic costs.

19. Assume that the production function is $Q = 2L^{\frac{1}{2}} + 3K$ and that the input demand for both capital and labor is positive.

(a) Set up the objective function for finding the cost function.

$$\min_{L,K} \max_{\lambda} wL + rK - \lambda \left(2\sqrt{L} + 3K - Q\right)$$

(b) Find the first order conditions of the objective function you just set up.

$$w - \frac{\lambda}{\sqrt{L}} = 0$$
$$r - \lambda = 0$$
$$2\sqrt{L} + 3K - Q = 0$$

(c) Solve for the buck for the bang for labor and capital.

$$\lambda = w\sqrt{L}$$
$$\lambda = \frac{r}{3}$$

(d) By equalizing the buck for the bang conditions find the input demand for Labor.

$$w\sqrt{L} = \frac{r}{3}$$
$$L = \left(\frac{r}{3w}\right)^2$$

(e) From the production constraint find the input demand for capital.

$$2\sqrt{L} + 3K - Q = 0$$
$$2\sqrt{\left(\frac{r}{3w}\right)^2} + 3K - Q = 0$$
$$K = \frac{1}{3}Q - \frac{2}{9}\frac{r}{w}$$

(f) Find the cost function.

$$C(w, r, Q) = w\frac{1}{9}\frac{r^2}{w^2} + r\left(\frac{1}{3}Q - \frac{2}{9}\frac{r}{w}\right)$$
$$C(w, r, Q) = \frac{1}{3}rQ - \frac{1}{9w}r^2$$

(g) Explain the general envelope theorem. You may use the cost function above to illustrate the concept.

The general envelope theorem tells us that when taking derivatives of an optimized function all of the indirect effects disappear. This is because the marginal benefit and marginal cost of these indirect effects is already balanced out. To illustrate this with the cost function above:

$$C(w, r, Q) \min_{L,K} \max_{\lambda} wL + rK - \lambda\left(2\sqrt{L} + 3K - Q\right)$$

$$\frac{\partial C}{\partial w} = L + \frac{\partial L}{\partial w}\left(w - \frac{\lambda}{\sqrt{L}}\right) + \frac{\partial K}{\partial w}(r - \lambda 3) + \frac{\partial \lambda}{\partial w}\left(2\sqrt{L} + 3K - Q\right)$$

but as we showed in part (b) $\left(w - \frac{\lambda}{\sqrt{L}}\right) = (r - \lambda 3) = \left(2\sqrt{L} + 3K - Q\right) = 0$ *at the optimal* $\{L, K, \lambda\}$.

(h) Using the envelope theorem and the cost function found in part *(f)* find the demand for labor and capital. You will get partial credit for this question even if you did not find the cost function in *(f)*

$$C(w,r,Q) = \frac{1}{3}rQ - \frac{1}{9w}r^2$$
$$\frac{\partial C}{\partial w} = \frac{1}{9}\frac{r^2}{w^2}$$
$$\frac{\partial C}{\partial r} = \frac{Q}{(3)} - \frac{2}{9}\frac{r}{w}$$

20. Prove that the Cost function is concave in input prices. Or: $C(\alpha w + (1-\alpha)\tilde{w}, \alpha r + (1-\alpha)\tilde{r}, Q) \geq \alpha C(w,r,Q) + (1-\alpha)C(\tilde{w},\tilde{r},Q)$.

    (a) In intuitive terms explain why this relationship should hold.

    On the left hand side of this equation the firm basically has one choice, how to produce at the prices $\{\alpha w + (1-\alpha)\tilde{w}, \alpha r + (1-\alpha)\tilde{r}\}$. *On the right hand side they have two choices, how to produce at* $\{w,r\}$ *and how to produce at* $\{\tilde{w},\tilde{r}\}$. *Since more choice is always better we get the inequality.*

    (b) Let $\{L^*,K^*\}$ be cost minimizing at $\{w,r\}$ and $\{L_\alpha,K_\alpha\}$ be cost minimizing at $\{\alpha w + (1-\alpha)\tilde{w}, \alpha r + (1-\alpha)\tilde{r}\}$. Is $wL_\alpha + rK_\alpha \geq wL^* + rK^*$, $wL_\alpha + rK_\alpha \leq wL^* + rK^*$, or can we not tell? Why is this?

    $wL_\alpha + rK_\alpha \geq wL^* + rK^*$ *because this is what it means for* $\{L^*,K^*\}$ *to be cost minimizing at* $\{w,r\}$.

    (c) Using algebra show that: $C(\alpha w + (1-\alpha)\tilde{w}, \alpha r + (1-\alpha)\tilde{r}, Q) = \alpha(wL_\alpha + rK_\alpha) + (1-\alpha)(\tilde{w}L_\alpha + \tilde{r}K_\alpha)$

$$\begin{aligned}
&C(\alpha w + (1-\alpha)\tilde{w}, \alpha r + (1-\alpha)\tilde{r}, Q) = \\
&(\alpha w + (1-\alpha)\tilde{w})L_\alpha + (\alpha r + (1-\alpha)\tilde{r})K_\alpha = \\
&\alpha w L_\alpha + (1-\alpha)\tilde{w}L_\alpha + \alpha r K_\alpha + (1-\alpha)\tilde{r}K_\alpha = \\
&\alpha w L_\alpha + \alpha r K_\alpha + (1-\alpha)\tilde{w}L_\alpha + (1-\alpha)\tilde{r}K_\alpha = \\
&\qquad \alpha(wL_\alpha + rK_\alpha) + (1-\alpha)(\tilde{w}L_\alpha + \tilde{r}K_\alpha)
\end{aligned}$$

    (d) Combine the insights of the last two parts to prove the statement.

$$\begin{aligned}
C(\alpha w + (1-\alpha)\tilde{w}, \alpha r + (1-\alpha)\tilde{r}, Q) &= \alpha(wL_\alpha + rK_\alpha) + (1-\alpha)(\tilde{w}L_\alpha + \tilde{r}K_\alpha) \\
&\geq \alpha(wL^* + rK^*) + (1-\alpha)(\tilde{w}L_\alpha + \tilde{r}K_\alpha) \\
&= \alpha C(w,r,Q) + (1-\alpha)(\tilde{w}L_\alpha + \tilde{r}K_\alpha)
\end{aligned}$$

    of course by the insight in part (b) we also know that

$$\tilde{w}L_\alpha + \tilde{r}K_\alpha \geq \tilde{w}\tilde{L} + \tilde{r}\tilde{K} = C(\tilde{w},\tilde{r},Q)$$

thus

$$\alpha C(w,r,Q)+(1-\alpha)(\tilde{w}L_\alpha+\tilde{r}K_\alpha)\geq \alpha C(w,r,Q)+(1-\alpha)C(\tilde{w},\tilde{r},Q)$$

and the property is proven.

21. Assume that the production function is $Q=\frac{5}{2}L^{\frac{1}{2}}+K$ and that the input demand for both labor and capital is positive.

    (a) Solve for the short run cost function.

$$\frac{5}{2}\sqrt{L}+K-Q=0$$
$$\frac{5}{2}\sqrt{L}=Q-K$$
$$L^{\frac{1}{2}}=\frac{Q-K}{\frac{5}{2}}$$
$$L=\frac{4}{25}(Q-K)^2$$

$$C^{SR}(w,r,K,Q)=w\frac{4}{25}(Q-K)^2+rK$$

    (b) Let $g(r,w,K)=\min_{L,\lambda} f(L,\lambda,r,w,K)$ and $\{L(w,r,K),\lambda(r,w,K)\}$ be the minimizing values of the right hand side, or $g(r,w,K)=f(L(w,r,K),\lambda(r,w,K),r,w,K)$. State and prove the general envelope theorem using this function. What does this function tell us about the short run cost function when $\frac{\partial C^{SR}(w,r,K)}{\partial K}=0$?
    The general envelope theorem says that $\frac{\partial g}{\partial r}=\frac{\partial f}{\partial r}$, $\frac{\partial g}{\partial w}=\frac{\partial f}{\partial w}$, $\frac{\partial g}{\partial K}=\frac{\partial f}{\partial K}$. *To prove this you take the derivative with respect to any of these variables:*

$$\frac{\partial g}{\partial r}=\frac{\partial f}{\partial r}+\frac{\partial f}{\partial L}\frac{\partial L}{\partial r}+\frac{\partial f}{\partial \lambda}\frac{\partial \lambda}{\partial r}$$

    and note that do to minimization $\frac{\partial f}{\partial L}=\frac{\partial f}{\partial \lambda}=0$.
    This tells us that $\frac{\partial C^{SR}(w,r,K)}{\partial K}=r-\lambda f_k$ *so when* $\frac{\partial C^{SR}(w,r,K)}{\partial K}=0$ *this is the optimal amount of capital.*

    (c) Use the envelope theorem to find the long run cost function.

$$\frac{\partial C^{SR}(w,r,K,Q)}{\partial K}=-\frac{8}{25}w(Q-K)+r=0$$

$$r=\frac{8}{25}w(Q-K)$$
$$K=Q-\frac{25}{8}\frac{r}{w}$$

:

$$\begin{aligned} C^{LR}(w,r,K,Q) &= C^{SR}(w,r,K^*,Q) \\ &= w\left(\frac{Q-(Q-\frac{25}{8}\frac{r}{w})}{\frac{5}{2}}\right)^2 + r\left(Q-\frac{25}{8}\frac{r}{w}\right) \\ &= Qr - \frac{25}{16}\frac{r^2}{w} \end{aligned}$$

(d) Using the envelope theorem and the cost functions you found in part a and c find the demand for labor. You will get partial credit for explaining how to do this even if you did not find the cost functions.
According to the envelope theorem:

$$\frac{\partial C^{SR}}{\partial w} = L^{SR}$$
$$\frac{\partial C^{LR}}{\partial w} = L^{LR}$$

$$\frac{\partial C^{SR}}{\partial w} = \frac{4}{25}(Q-K)^2$$

and this is obviously the short run demand for labor.

$$\frac{\partial C^{LR}}{\partial w} = \frac{25}{16}\frac{r^2}{w^2} = L^{LR}$$

22. If a firm has a cost function $c(w,r,Q)$ show that it is non-decreasing in input prices. Or that if $w > \tilde{w}$ and $r > \tilde{r}$ then $c(w,r,Q) \geq c(\tilde{w},\tilde{r},Q)$. Notice you can not make any assumptions about the production function.

    (a) If $\{L^*, K^*\}$ are cost minimizing at the prices $\{w, r\}$ show that producing using $\{L^*, K^*\}$ costs less at the input prices $\{\tilde{w}, \tilde{r}\}$.
    $wL^* + rK^* \geq \tilde{w}L^* + \tilde{r}K^*$

    (b) If $\left\{\tilde{L}, \tilde{K}\right\}$ are cost minimizing at the prices $\{\tilde{w}, \tilde{r}\}$ show a relationship between the costs of producing using $\left\{\tilde{L}, \tilde{K}\right\}$ and the costs of producing using $\{L^*, K^*\}$ at the prices $\{\tilde{w}, \tilde{r}\}$.
    since $\left\{\tilde{L}, \tilde{K}\right\}$ are cost minimizing at the prices $\{\tilde{w}, \tilde{r}\}$ any other combination of inputs at this price setting would cost more.

    (c) Use the previous two arguments to conclude that $c(w,r,Q) \geq c(\tilde{w},\tilde{r},Q)$
    $c(w,r,Q) = wL^* + rK^* \geq \tilde{w}L^* + \tilde{r}K^* \geq \tilde{w}\tilde{L} + \tilde{r}\tilde{K} = c(\tilde{w},\tilde{r},Q)$

23. What is the difference between accountants and economists? Name and define a cost that economists think is a cost and accountants don't, and a cost accountants think is a cost and economists don't.

Accountants are focused on what happened in the past, economists advise decision makers and are focused on the future. Economists think opportunity costs—the value of a good on the marketplace—are important but accountants think they are ridiculous. Accountants value sunk costs—a cost which is not recoverable—while economists ignore them.

24. Define Fixed Start Up costs. Are they part of variable costs? What is the difference between these costs and Fixed Sunk costs?

    A fixed start up cost is any cost that must be paid if output is greater than zero but does not have to be paid if output is zero. Yes they are part of variable costs, and the difference between these and Fixed Sunk costs is that fixed sunk costs have to be paid even if output is zero.

25. Let a firm's average costs be $\frac{C(q)}{q}$, and their marginal costs be $\frac{dC(q)}{dq}$. Prove that this firm's average costs equal their marginal costs at the quantity that minimizes average costs.

    (a) Set up the objective function you would use to minimize average costs.

    $$\min_q \frac{C(q)}{q}$$

    (b) Find the first order condition.

    $$\frac{dC(q)}{dq}\frac{1}{q} - \frac{C(q)}{q^2} = 0$$

    (c) Find a term in the first order condition that is equal to marginal costs, and another term that is equal to average costs.

    $$MC\frac{1}{q} - AC\frac{1}{q} = 0$$

    (d) With some algebra prove the statement above.
    Multiplying by q does not change the equation, so we have

    $$MC - AC = 0$$
    $$MC = AC$$

26. Assume that a firm has a short run cost function given by $C^{SR}(w, r, K, Q) = \frac{1}{4}K^2 - 3\frac{QK}{rw}$

    Before answering this question let me point out that this is not a cost function, so the answers will be a little bit funky. You can see it is not a cost function because it is not homogenous of degree one in input prices. Also when one finds the long run cost function it is negative, another problem.

(a) Find the firm's short run demand for labor using the envelope theorem. (Remember that w is the wage, or the cost of a unit of labor, and that this should be a function of w, r, K, and Q).

$$\frac{\partial C^{SR}}{\partial w} = 3\frac{QK}{rw^2}$$

(b) Find the firm's long run cost function by using the envelope theorem

$$\frac{\partial C^{SR}}{\partial K} = 2\left(\frac{1}{4}\right)K - 3\frac{Q}{rw} = 0$$
$$K = 6\frac{Q}{rw}$$

$$C^{LR}(w,r,Q) = \frac{1}{4}K^{*2} - 3\frac{QK^*}{rw}$$
$$C^{LR}(w,r,Q) = \frac{1}{4}\left(6\frac{Q}{rw}\right)^2 - 3\frac{Q\left(6\frac{Q}{rw}\right)}{rw}$$
$$= -9\frac{Q^2}{r^2w^2}$$

which is clearly impossible since the marginal cost is negative, sigh.

(c) Define the general envelope theorem. *The general envelope theorem says that whenever you have an optimized function—like a cost function—the only impact of control variables—like w, r, K and Q—is the direct effect. The indirect effects disappear. Thus we know that* $\frac{\partial C^{SR}}{\partial w} = L$ *and* $\frac{\partial C^{SR}}{\partial K} = r - \lambda MP_k$.

27. Assume that you produce output, Q, with two inputs: capital (K) and labor (L). The price of a unit of capital is r and the price of a unit of labor is w. Assume that $\{L, K\}$ is cost minimizing at the input prices $\{w, r\}$ and that $\left\{\hat{L}, \hat{K}\right\}$ is cost minimizing at $\{\hat{w}, \hat{r}\}$.

(a) What is the relationship between the cost of producing using inputs $\{L, K\}$ and using inputs $\left\{\hat{L}, \hat{K}\right\}$ at the prices $\{\hat{w}, \hat{r}\}$? Is the cost of using $\{L, K\}$ less than or more than the cost of using $\left\{\hat{L}, \hat{K}\right\}$?

$\hat{w}L + \hat{r}K \geq \hat{w}\hat{L} + \hat{r}\hat{K}$

since $\left\{\hat{L}, \hat{K}\right\}$ *is cost minimizing at* $\{\hat{w}, \hat{r}\}$ *any other input combination at the prices* $\{\hat{w}, \hat{r}\}$ *would have a higher cost.*

(b) Assume that $\hat{w} \leq w$ and that $\hat{r} \leq r$. Prove that $C(w,r,Q) \geq C(\hat{w}, \hat{r}, Q)$, or that the cost function is non-decreasing in input prices.

$C(w,r,Q) = wL + rK \geq \hat{w}L + \hat{r}K \geq \hat{w}\hat{L} + \hat{r}\hat{K} = C(\hat{w}, \hat{r}, Q)$

28. Assume that the production function is $Q = 2L^{\frac{1}{2}} + K$ and that $Q > K$ for parts a through f.

(a) Set up the objective function you need to solve in order to find the short run cost function (where capital (K) is fixed).

$$\min_{L} \max_{\lambda} wL + rK - \lambda \left(2L^{\frac{1}{2}} + K - Q\right)$$

(b) Find the first order conditions of the objective function you just set up.

$$w - \lambda L^{-\frac{1}{2}} = 0$$
$$2L^{\frac{1}{2}} + K - Q = 0$$

(c) Explain why you do not need to actually find the first order conditions to find the short run cost function. *Because the production constraint* $(Q = 2L^{\frac{1}{2}} + K)$ *completely determines* L.

(d) Find the short run cost function.

$$L = \left(\frac{Q-K}{2}\right)^2$$
$$C(Q, w, r, K) = w\left(\frac{Q-K}{2}\right)^2 + rK$$

(e) Explain what the envelope theorem tells us, you may use the short run cost function to illustrate the concept.. *In general if*

$$C(Q, w, r, w_m) = \min wL + rK + w_m M \text{ s.t. } Q = f(K, L, M)$$

or

$$\pi(p, w, r, w_m) = \max pf(K, L, M) - (wL + rK + w_m M)$$

or any other such function then:

$$\frac{\partial C}{\partial w} = L$$

in general

$$\frac{\partial C}{\partial w} = L + \frac{\partial L}{\partial w}(w - MP_L) + \frac{\partial K}{\partial w}(r - MP_K) + \frac{\partial M}{\partial w}(w_m - MP_M)$$

but by optimization we know that $(r - MP_K) = 0$ *and thus all of the indirect effects drop out.*

(f) Find the derivative of the short run cost function with respect to capital, K.

$$\frac{\partial C(Q, w, r, K)}{\partial K} = -w\left(\frac{Q-K}{2}\right) + r$$

(g) Find the long run cost function, you can only get half credit if you do not use the envelope theorem. (You can assume that the optimal amount of labor and capital are both positive.)

$$\frac{\partial C(Q,w,r,K)}{\partial K} = -w\left(\frac{Q-K}{2}\right) + r = 0$$

$$\frac{2r - wQ}{w} = K$$

$$C(Q,w,r) = w\left(\frac{Q - \frac{2r-wQ}{w}}{2}\right)^2 + r\frac{2r-wQ}{w}$$

$$= \frac{Q^2w^2 - 3Qrw + 3r^2}{w}$$

29. If a firms production function is:

$$Q = (L+6)^{\frac{1}{3}} K^{\frac{2}{3}}$$

(a) Set up and solve the cost minimization problem, where the price of L (labor) is w and the price of K (capital) is r.
ASSUME THAT BOTH L AND K ARE DEMANDED IN STRICTLY POSITIVE QUANTITIES. (Or the solution is interior.)

i. Set up the Lagrangian.

$$\min_{L,K,\lambda} wL + rK + \lambda\left(Q - (L+6)^{\left(\frac{1}{3}\right)} K^{\left(\frac{2}{3}\right)}\right)$$

ii. Solve for the first order conditions.

$$w - \lambda\frac{1}{3}\frac{Q}{(L+6)} = 0$$

$$r - \lambda\frac{2}{3}\frac{Q}{K} = 0$$

$$Q - (L+6)^{\frac{1}{3}} K^{\frac{2}{3}} = 0$$

iii. Solve for the “buck for the bang” for both Labor and Capital.

$$\lambda = \frac{3w}{Q}(L+6)$$

$$\lambda = \frac{3}{2}\frac{r}{Q}K$$

iv. Solve for K in terms of L and input prices.

$$\frac{3}{2}\frac{r}{Q}K = \frac{3w}{Q}(L+6)$$

$$K = 2\frac{w}{r}(L+6)$$

v. Solve for the demand curve for L using the production constraint

$$Q - (L + (6))^{\left(\frac{1}{3}\right)} \left(2\frac{w}{r}(L+6)\right)^{\left(\frac{2}{3}\right)} = 0$$
$$\frac{Q}{\left(2\frac{w}{r}\right)^{\left(\frac{2}{3}\right)}} - (6) = L$$

vi. Solve for the demand curve for K

$$K = \left(2\frac{w}{r}\right)^{\frac{1}{3}} Q$$

(b) If $r = 1$ find a condition on w such that the firm will use only capital in production.

This is a corner solution which means the buck for the bang for labor must always be higher than the buck for the bang for capital.

$$\frac{r}{MP_K} \leq \frac{w}{MP_L}$$
$$\frac{3}{2}\frac{r}{Q}K \leq 3\frac{w}{Q}(L+6)$$

for all $L > 0$ *or*

$$\frac{3}{2}K < 18w$$
$$\frac{1}{12}K < w$$

30. Assume that you have a short run cost function:

$$C^{SR}(w, r, Q, K) = \frac{wQ^3}{K^{\frac{2}{3}}} + rK$$

(a) Find the marginal cost of the short run cost function, and establish whether it is increasing, decreasing or constant.

$$MC = 3\frac{w}{K^{\left(\frac{2}{3}\right)}}Q^2$$
$$\frac{\partial MC}{\partial Q} = 6\frac{w}{K^{\left(\frac{2}{3}\right)}}Q \geq 0 \text{ increasing}$$

(b) Write down and explain the general envelope theorem.

For an optimized function (like the cost function) the derivative with regards to exogenous variables (like $\{w, r, Q, K\}$ *above) are the direct effects alone. The effects on the choice variables (*L *above) will be washed out because the marginal benefit and the marginal cost of these choice variables will be equal.*

(c) Using the envelope theorem solve for the input demand for capital and the long run cost function.

$$\frac{\partial C^{SR}(w,r,Q,K)}{\partial K} = -\frac{2}{3}\frac{wQ^3}{K^{\frac{5}{3}}} + r = 0$$

$$K = \left(\frac{2}{3}\right)^{\frac{3}{5}}\left(\frac{w}{r}\right)^{\frac{3}{5}} Q^{\frac{9}{5}}$$

$$C^{LR}(w,r,Q) = \frac{wQ^{(3)}}{\left(\left(\frac{2}{3}\right)^{\frac{3}{5}}\left(\frac{w}{r}\right)^{\frac{3}{5}} Q^{\frac{9}{5}}\right)^{\left(\frac{2}{3}\right)}} + r\left(\frac{2}{3}\right)^{\frac{3}{5}}\left(\frac{w}{r}\right)^{\frac{3}{5}} Q^{\frac{9}{5}}$$

$$= \left(\left(\frac{2}{3}\right)^{-\frac{2}{5}} + \left(\frac{2}{3}\right)^{\frac{3}{5}}\right) Q^{\frac{9}{5}} r^{\frac{2}{5}} w^{\frac{3}{5}}$$

31. If a firm has the production function $Q = \ln L + 8\ln K$ and the price of labor is w and the price of capital is r.

(a) Find the short run cost function (where K is fixed.)

$$\min_{l} wL + rK \text{ s.t. } Q = \ln L + 8\ln K$$

$$L = \frac{e^Q}{K^8}$$

$$C(Q,w,r,K) = w\frac{e^Q}{K^8} + rK$$

(b) Explain the *general* envelope theorem. How can this be applied to find the long run cost function from the short run cost function? *In general if*

$$C(Q,w,r,w_m) = \min wL + rK + w_m M \text{ s.t. } Q = f(K,L,M)$$

or

$$\pi(p,w,r,w_m) = \max pf(K,L,M) - (wL + rK + w_m M)$$

or any other such function then:

$$\frac{\partial C}{\partial w} = L$$

in general

$$\frac{\partial C}{\partial w} = L + \frac{\partial L}{\partial w}(w - MP_L) + \frac{\partial K}{\partial w}(r - MP_K) + \frac{\partial M}{\partial w}(w_m - MP_M)$$

but by optimization we know that $(r - MP_K) = 0$ and thus all of the indirect effects drop out.

This is useful to find the long run cost function because it tells us that;

$$\frac{\partial C(Q,w,r,K)}{\partial K} = r - MP_K$$

or all of the indirect effects of changing capital (in this case changing labor is the only one) drop out. Thus when $\frac{\partial C(Q,w,r,K)}{\partial K} = 0$ we know we are at the optimal level of capital, or we have found the long run value of capital for that $\{Q,w,r\}$

(c) Using the envelope theorem find the long run cost function. (Note you should use the envelop theorem, you will only receive minor credit for deriving it directly.)

$$\begin{aligned}
\frac{\partial C(Q,w,r,K)}{\partial K} &= -8w\frac{e^Q}{K^9} + r = 0 \\
\left(\frac{8we^Q}{r}\right)^{\frac{1}{9}} &= K \\
C(Q,w,r) &= w\frac{e^Q}{\left(\left(\frac{8we^Q}{r}\right)^{\frac{1}{9}}\right)^8} + r\left(\frac{8we^Q}{r}\right)^{\frac{1}{9}} \\
&= \frac{9}{8^{\frac{2}{3}}}w^{\frac{1}{9}}r^{\frac{8}{9}}e^{\frac{1}{9}Q}
\end{aligned}$$

32. If your production function is $Q = f(L,K)$, the price of labor is w and the price of capital is r prove that if L^* is the cost minimizing demand given $\{Q,w,r\}$ then:

$$\frac{\Delta L^*}{\Delta w} \leq 0$$

Note you can only assume this person is a rational cost minimizer.

Assume that $w' > w$ and that $\{L',K'\}$ are cost minimizing at $\{w',r\}$ and that $\{L,K\}$ are cost minimizing at $\{w,r\}$ then we know:

$$\begin{aligned}
w'L' + rK' &\leq w'L + rK \\
wL' + rK' &\geq wL + rK
\end{aligned}$$

or

$$-(wL' + rK') \leq -(wL + rK)$$

and thus we know that

$$
\begin{aligned}
w'L' + rK' + -(wL' + rK') &\leq w'L + rK - (wL + rK) \\
w'L' - wL' &\leq w'L - wL \\
(w' - w)L' &\leq (w' - w)L \\
(w' - w)L' - (w' - w)L \leq 0 \\
(w' - w)(L' - L) \leq 0
\end{aligned}
$$

and since $\Delta w = w' - w > 0$ *we know that*

$$
\begin{aligned}
\frac{(w' - w)(L' - L)}{\Delta w} &\leq \frac{0}{\Delta w} \\
\frac{\Delta L}{\Delta w} &\leq 0
\end{aligned}
$$

$\Delta L = L' - L$ *as required.*

33. Guclu runs a cloth dyeing plant with three main inputs, labor—L, materials (cloth)—M, and capital (the physical plant)—K. His capital is fixed at $\bar{K}$. The cost of labor is w, of materials is w_m and the capital costs r dollars. His production function is:

$$Q = L^{\frac{1}{2}} M^{\frac{1}{4}} K^{\frac{1}{3}}$$

(a) Set up his short run cost minimization problem, i.e. his Lagrangian.

$$\min_{L,M} wL + w_m M + rK + \lambda \left(Q - L^{\frac{1}{2}} M^{\frac{1}{4}} K^{\frac{1}{3}} \right)$$

(b) Solve for his short run cost function.

$$
\begin{aligned}
w - \lambda \frac{1}{2} L^{\frac{1}{2}-1} M^{\frac{1}{4}} K^{\frac{1}{3}} &= 0 \\
w - \lambda \frac{1}{2} \frac{L^{\frac{1}{2}} M^{\frac{1}{4}} K^{\frac{1}{3}}}{L} &= 0 \\
w - \lambda \frac{1}{2} \frac{Q}{L} &= 0 \\
w_m - \lambda \frac{1}{4} \frac{Q}{M} &= 0 \\
Q - L^{\frac{1}{2}} M^{\frac{1}{4}} K^{\frac{1}{3}} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{w}{\frac{1}{2}\frac{Q}{L}} = \lambda &= \frac{w_m}{\frac{1}{4}\frac{Q}{M}} \\
\frac{w}{\frac{1}{2}\frac{Q}{L}} &= \frac{w_m}{\frac{1}{4}\frac{Q}{M}} \\
L &= 2\frac{w_m}{w} M
\end{aligned}
$$

$$
\begin{aligned}
c\left(w, w_m, K\right) &= wL^* + w_m M^* + rK \\
&= w\left(2\frac{w_m}{w}M^*\right) + w_m M^* + rK \\
&= 3w_m M^* + rK
\end{aligned}
$$

which is a pretty cool trick you must admit. We also have:

$$
\begin{aligned}
Q &= L^{\frac{1}{2}} M^{\frac{1}{4}} K^{\frac{1}{3}} \\
&= \left(2\frac{w_m}{w}M\right)^{\frac{1}{2}} M^{\frac{1}{4}} K^{\frac{1}{3}} \\
&= \left(2\frac{w_m}{w}\right)^{\frac{1}{2}} M^{\frac{1}{4}+\frac{1}{2}} K^{\frac{1}{3}} \\
\frac{Q}{K^{\frac{1}{3}}}\left(\frac{w}{2w_m}\right)^{\frac{1}{2}} &= M^{\frac{3}{4}} \\
\left(\frac{Q}{K^{\frac{1}{3}}}\left(\frac{w}{2w_m}\right)^{\frac{1}{2}}\right)^{\frac{4}{3}} &= M
\end{aligned}
$$

$$
\begin{aligned}
c\left(w, w_m, K\right) &= 3w_m\left(\frac{Q}{K^{\frac{1}{3}}}\left(\frac{w}{2w_m}\right)^{\frac{1}{2}}\right)^{\frac{4}{3}} + rK \\
&= \frac{3}{2^{\frac{2}{3}}} w_m^{\frac{1}{3}} w^{\frac{2}{3}} \frac{Q^{\frac{4}{3}}}{K^{\frac{4}{9}}} + rK
\end{aligned}
$$

(c) Using the envelope theorem find his optimal level of capital and his long run cost function.

$$
\begin{aligned}
\frac{\partial c\left(w, w_m, K\right)}{\partial K} &= -\frac{4}{9}\frac{3}{2^{\frac{2}{3}}} w_m^{\frac{1}{3}} w^{\frac{2}{3}} \frac{Q^{\frac{4}{3}}}{K^{\frac{4}{9}+1}} + r = 0 \\
r &= \frac{4}{9}\frac{3}{2^{\frac{2}{3}}} w_m^{\frac{1}{3}} w^{\frac{2}{3}} \frac{Q^{\frac{4}{3}}}{K^{\frac{13}{9}}} \\
K^{\frac{13}{9}} &= \frac{4}{9}\frac{3}{2^{\frac{2}{3}}} \frac{w_m^{\frac{1}{3}} w^{\frac{2}{3}}}{r} Q^{\frac{4}{3}} \\
K &= \left(\frac{4}{9}\frac{3}{2^{\frac{2}{3}}} \frac{w_m^{\frac{1}{3}} w^{\frac{2}{3}}}{r} Q^{\frac{4}{3}}\right)^{\frac{9}{13}} \\
&= \left(\frac{4}{9}\frac{3}{2^{\frac{2}{3}}}\right)^{\frac{9}{13}} \frac{w_m^{\frac{3}{13}} w^{\frac{6}{13}}}{r^{\frac{9}{13}}} Q^{\frac{12}{13}}
\end{aligned}
$$

$$
\begin{aligned}
c(w, w_m, K) &= \frac{3}{2^{\frac{2}{3}}} w_m^{\frac{1}{3}} w^{\frac{2}{3}} \frac{Q^{\frac{4}{3}}}{K^{\frac{4}{9}}} + rK \\
&= \frac{3}{2^{\frac{2}{3}}} w_m^{\frac{1}{3}} w^{\frac{2}{3}} \frac{Q^{\frac{4}{3}}}{\left(\left(\frac{4}{9}\frac{3}{2^{\frac{2}{3}}}\right)^{\frac{9}{13}} \frac{w_m^{\frac{3}{13}} w^{\frac{6}{13}}}{r^{\frac{9}{13}}} Q^{\frac{12}{13}}\right)^{\frac{4}{9}}} + r\left(\frac{4}{9}\frac{3}{2^{\frac{2}{3}}}\right)^{\frac{9}{13}} \frac{w_m^{\frac{3}{13}} w^{\frac{6}{13}}}{r^{\frac{9}{13}}} Q^{\frac{12}{13}} \\
&= r^{\frac{4}{13}} w_m^{\frac{3}{13}} w^{\frac{6}{13}} Q^{\frac{12}{13}} (2.8803)
\end{aligned}
$$

(d) Explain the envelope theorem using cost minimization as an example. Why is it a surprising result and why is it true?

The cost function can be written as:

$$C(w, r) = \min_{L,K} wL + rK + \lambda (Q - f(L, K))$$

and using this we can see that the effect of changing w *on the costs is:*

$$\frac{\partial C}{\partial w} = L + w\frac{\partial L}{\partial w} + r\frac{\partial K}{\partial w} - \lambda\frac{\partial f}{\partial L}\frac{\partial L}{\partial w} - \lambda\frac{\partial f}{\partial K}\frac{\partial K}{\partial w}$$

which is an awful mess. Basically this summarizes all direct and indirect affects of raising the wage. But the envelope theorem points out we can simplify this dramatically:

$$
\begin{aligned}
\frac{\partial C}{\partial w} &= L \\
&+ \left(w - \lambda\frac{\partial f}{\partial L}\right)\frac{\partial L}{\partial w} \\
&+ \left(r - \lambda\frac{\partial f}{\partial K}\right)\frac{\partial K}{\partial w}
\end{aligned}
$$

and by optimization, $w - \lambda\frac{\partial f}{\partial L} = 0$ *and* $r - \lambda\frac{\partial f}{\partial K} = 0$ *or* $\frac{\partial C}{\partial w} = L$*! That's pretty surprising, you must admit. Why is it true? Because optimization tells us that the marginal cost (*w*) and the marginal benefit (*$\lambda\frac{\partial f}{\partial L}$*) must be balanced out, and this means that any indirect affect (like the affect from altering the amount of labor we demand) will be wiped out.*

34. Why do economists and accountant differ on what is, and what is not a cost? What major class of costs do accountants consider not costs? What major class of accounting costs do economists consider not costs?

It is essentially because accountants want to record what happened, their job is to find out if reports are accurate. Thus they are concerned with the past. Economists only care about the future, because they want to help people make decisions.

For example, Accountants don't think that opportunity costs are actually costs, because they are related to someone's best outside opportunity and didn't actually happen. Accountants think that sunk costs—money that has already been lost paying for a project—should be counted as costs in the future, but economists do not think so, because they've already been spent, and thus unimportant for future decisions.

35. Consider the two cost functions

$$C_1(q) = 25 + \frac{1}{9}q^2$$
$$C_2(q) = 16 + 2q$$

(a) If I tell you one is a long run cost function and one is a short run cost function associated with the same technology, which one is the long run cost function? And why?

The relationship we know between long run and short run costs are that in the long run costs MUST be lower. Hence let us check the sign of $C_1(q) - C_2(q)$:

$$C_1(q) - C_2(q) = 25 + \frac{1}{9}q^2 - 16 - 2q = 9 + \frac{1}{9}q^2 - 2q = \frac{1}{9}(q-9)^2 \geq 0$$

So we can conclude that $C_1(q)$ is short run cost function and $C_2(q)$ is the long run.

(b) What important result in producer theory is illustrated by the above?

Note that the only point at which $C_1(q) = C_2(q)$ is $q = 9$. In fact, at this level of q the long run cost minimizing level of the input which was fixed in the short run is equal to the fixed short run amount. For every other value of q, since long run optimum level of the input is different (less) than the amount which is fixed in the short run, long run cost is strictly less than the short run at all such q's. This shows that geometrically, the long run cost function is the lower envelope of the short run cost functions corresponding to different levels of the fixed input.

36. Henry's Hippo Hippodrome (visualize it) uses 3 gardeners for each lawn mower they buy. The gardeners cost \$60 a day and lawn mowers rent for \$10 a day. If the MP of a gardener is 80 and of the lawn mower are 30 are they minimizing their cost? If not, how should they change their usage of gardeners and lawn mowers?

Cost minimization requires the employment of the factors until the marginal product per dollar is equal among all.

$$\frac{\text{MP gardener}}{\text{cost of gardener}} = \frac{80}{60} = \frac{8}{6}$$

$$\frac{\text{MP lawnmower}}{\text{cost of lawnmower}} = \frac{30}{10} = 3$$

Since they are not equal, we conclude that costs are not minimized. The MP/cost of gardener (bang for buck) is lower than for a lawnmower, we should increase the number of lawnmowers and decrease the number of gardeners employed.

37. The production function of firm A is $Q = 10K^{.5}L^{.5}$ and of it's competitor (firm B) is $Q = 10K^{.2}L^{.8}$.

 (a) In the long run show that if they both use the same level of capital and labor ($K = L$) then they are equally efficient competitors.

 Plugging in $K = L$ into the production of the both firms, we have

$$Q^A = 10L^{0.5}L^{0.5} = 10L$$

$$Q^B = 10L^{0.2}L^{0.8} = 10L$$

 which implies that firm A, as well as firm B, by employing L units of labor and L units of capital produce $10L$ units of output. Note that firm A employs the same amount of labor, L, as firm B; and also the same amount of capital, L, as firm B. And at the end the amount they produce is the same, $10L$; that is, they are equally efficient competitors.

 (b) In the short run, (when capital is fixed, and equal to one for both firms) which of them is a better competitor (has a higher marginal product)?

 Plugging $K = 1$ into the production function of the firms, we obtain

$$Q^A = 10L^{0.5} \qquad \text{and} \qquad Q^B = 10L^{0.8}$$

 The marginal products of labor in firm A and B are

$$\frac{\partial Q^A}{\partial L} = (10 * 0.5)L^{-0.5} = MPL^A$$

$$\frac{\partial Q^B}{\partial L} = (10 * 0.8)L^{-0.2} = MPL^B$$

 Then the ratio $\frac{MPL^A}{MPL^B}$ is equal to

$$\frac{MPL^A}{MPL^B} = \frac{5}{8L^{0.3}} < 1$$

 Inequality follows from the fact that L is a positive integer. Then we find $MPL^A < MPL^B$ and hence in short-run, when capital is fixed, firm B is more efficient than firm A.

38. You are in charge of cost control for a metropolitan bus company. A consultant finds that the cost of running a bus is independent of the number of people it carries, \$30 per run whether you have two people or fifty (the capacity). Thus during rush hour the average cost per passenger is 60 cents (when the buses are full) and off-hours the average cost \$1.67 because on average 18 people ride on each run. She suggests that you encourage more rush-hour business when costs are low, and discourage off-peak business. Do you agree with her? Discuss your answer.

    Suppose every hour, there is at least one bus running in a line regardless of whether there is a passenger or not. That is, there is a constant service provided which in turn implies that number of busses running in an hour can be increased or decreased but can not be canceled totally.

    If more rush hour business is encouraged, some of the off-peak passengers will switch to travelling at the peak-hours, making it necessary to add at least one more trip to the rush-hour, since all busses at this time are full. This leads a cost of \$30, and given our assumption this is an additional cost, that is the total cost increases by \$30. Depending on how many passengers have switched to the peak-hours the cost will change, for example if there are 55 more passengers this means two more busses will be necessary and this will increase the costs by \$60, since no busses have cancelled and two more busses are added to the line. The number of busses travelling at the off-peak hour is the same, according to our assumption and to the given information about the average number of passengers travelling in these hours. (It is given that there are on average 18 passengers which in turn implies that there is only one bus at those hours.)

    Given the above argument then the consultants advice should not be followed, but instead the off-peak businesses should be encouraged if there are more than one bus travelling at the rush-hour. If by encouraging off-peak businesses it is possible to make 50 people to travel in different off-peak hours it would be possible to decrease the number of rush-hour business without the need to increase the number of busses at the off-peak hours. For example if 20 people travelling in the rush-hour now travels at 2 o'clock, and other 20 at 3 o'clock, and the remaining 10 at 4 o'clock, then there would not be any need to increase the number of busses; but in fact one of the busses running at the rush-hour, say 6 o'clock, can be cancelled which will decrease the cost by \$30.

    The consultant has confused costs with demand. Cost of running a bus is the same in peak and off-peak hours. Demand is higher in peak hours. If we want to utilize our fleet more efficiently, we should encourage off-peak business in order to shift demand from peak to off-peak hours. If we manage to do that, we may be able to cancel some of the peak-hour busses without having to add new off-peak busses (since off-peak busses are running near empty right now)

3 Chapter 11—Profit Maximization

1. Assume that the cost function is $c(q) = 15q + 5q^3 + 32$, the fixed startup costs are $F_{st} = 10$.

   (a) Find the average total costs, average variable costs, and the marginal costs.

$$\begin{aligned} AC &= \frac{15q + 5q^3 + 32}{q} \\ AVC &= \frac{15q + 5q^3 + 32 - 22}{q} = \frac{1}{q}\left(10 + 5q^3 + 15q\right) \\ MC &= c'(q) = \frac{\partial c(q)}{\partial q} = 15 + 15q^2 \end{aligned}$$

   (b) Find the price at which the firm will shut down.

$$\begin{aligned} AVC(q_{sd}) &= MC(q_{sd}) \\ \frac{1}{q}\left(10 + 5q^3 + 15q\right) &= 15 + 15q^2 \\ 10 + 5q^3 + 15q &= 15q + 15q^3 \\ 10 &= 10q^3 \\ (1)^{\frac{1}{3}} &= 1 = q_{sd} \end{aligned}$$

$$p_{sd} = 15 + 15q_{sd}^2 = 15 + 15(1)^{\frac{2}{3}} = 15 + 15 = 30$$

   (c) Find the firm's supply curve.
   When the firm produces output:

$$\begin{aligned} p &= 15 + 15q^2 \\ p - 15 &= 15q^2 \\ \left(\frac{p - 15}{15}\right)^{\frac{1}{2}} &= q \end{aligned}$$

$$s(p) = \begin{cases} \left(\frac{p-15}{15}\right)^{\frac{1}{2}} & \text{if } \; p \geq 30 \\ 0 & \text{if } \; p \leq 30 \end{cases}$$

2. Let the total costs of a firm be $TC(Q) = c(Q) + F_{st} + F_{su}$ where $F_{st} > 0$ are fixed start up costs and $F_{su} > 0$ are fixed sunk costs. Assume that $c'(Q) > 0$ and that $c''(Q) \geq 0$.

   (a) Define marginal costs, average costs, and average variable costs for this general cost function.

$$MC = c'(Q)$$
$$AC = \frac{c(Q) + F_{st} + F_{su}}{Q}$$
$$AVC = \frac{c(Q) + F_{st}}{Q}$$

(b) In the graph below draw marginal costs, average costs, and average variable costs, make sure that the relationship between the functions is precise. If you do not trust your skills at drawing the relationship you may want to mention key points of your graph below.

Though it will be clear in my graph (since I am basing it on a real function) in general one needs to be careful to show that when $AC = MC$ *and* $AVC = MC$ *it is at the minimum of both of* AC *and* AVC *respectively.*

(c) In the graph below show the firm's supply curve.

Average costs are the highest u-shaped function, average variable costs are the lower u-shaped function, and marginal costs are the upward sloping line. We know that all of these curves must have these shapes because $c''(Q) \geq 0$. *The supply curve is the darkened line, it is 0 up to the P where AVC=MC and the line given by P=MC at that price and above.*

3. Hulusi's Hen House produces eggs (E) using the production function $E = 12H^8F$, if the price of a Hens (H) is w_h and the price of Feed (F) is w_f.

(a) Set up the Lagrangian and solve for the cost function.

$$\min_{H,F} \max_{\lambda} \pounds(H, F, \lambda) = w_h H + w_f F + \lambda(E - 12H^8 F)$$

The conditions for an interior optimal solution (H>0, F>0) to this problem are;

$$\begin{array}{rl} \frac{\partial \pounds}{\partial H} = w_h - 96\lambda F H^7 = 0 & w_h = 8\lambda E/H \\ \frac{\partial \pounds}{\partial F} = w_f - 12\lambda H^8 = 0 \implies & w_f = \lambda E/F \implies \\ \frac{\partial \pounds}{\partial \lambda} = E - 12H^8 F = 0 & E = 12H^8 F \end{array}$$

$$\begin{array}{r} \frac{H w_h}{8 w_f} = F \\ E = 12H^8 F \implies \\ = \frac{3}{2} H^9 \frac{w_h}{w_f} \end{array}$$

$$H^* = (E\frac{2}{3}\frac{w_f}{w_h})^{\frac{1}{9}}$$
$$F^* = \frac{(E\frac{2}{3}\frac{w_f}{w_h})^{\frac{1}{9}} w_h}{8w_f}$$

From this we can solve for the cost function:

$$\begin{aligned} c\,(E, w_h, w_f) &= w_h H^* + w_f F^* \\ &= w_h (E\frac{2}{3}\frac{w_f}{w_h})^{\frac{1}{9}} + w_f \frac{(E\frac{2}{3}\frac{w_f}{w_h})^{\frac{1}{9}} w_h}{8w_f} \\ &= AE^{\frac{1}{9}} \\ A &= \frac{3}{8} 2^{\frac{1}{9}} 3^{\frac{8}{9}} w_f^{\frac{1}{9}} w_h^{\frac{8}{9}} \end{aligned}$$

now obviously it takes some tricky algebra to figure out A (I used a computer program) but figuring out that the cost function is $E^{\frac{1}{9}}$ *times some function of prices is fairly easy.*

(b) Using the cost function find the profit function—be careful to check the second order conditions.
ANSWER:
So now our profit function is:

$$\pi\,(p, w_h, w_l) = \max_E pE - AE^{\frac{1}{9}}$$

the first order condition is:

$$p - \frac{1}{9} AE^{\frac{1}{9}-1} = 0$$

and the second order condition is:

$$-\frac{1}{9}\left(\frac{1}{9} - 1\right) AE^{\frac{1}{9}-2} = \frac{8}{81} AE^{\frac{1}{9}-2} > 0$$

and again with this production function we do not actually find the profit maximizing level of output, but instead the profit minimizing! If you think about it notice that:

$$p - \frac{1}{9}AE^{\frac{1}{9}-1} = p - \frac{\frac{1}{9}A}{E^{\frac{8}{9}}}$$

is increasing in E, and so if it's profitable to produce at all you should produce a large (infinite) level of output. But the key point is to notice how easy it was to check.

4. Assume that a firm has the production function

$$f(K, L) = K^{\frac{1}{2}}L^{\frac{3}{2}}$$

   (a) If $K = L$, and the wage is 3, then what is the price of capital (r)?

   First set up cost minimization problem, i.e. Lagrangian:

$$\min_{L,K} wL + rK + \lambda\left(Q - L^{\frac{3}{2}}K^{\frac{1}{2}}\right)$$

   Then the first order conditions give:

$$\frac{\partial \pounds}{\partial L} = w - \tfrac{3}{2}\lambda K^{\frac{1}{2}}L^{\frac{1}{2}} = 0 \Longrightarrow w = \tfrac{3}{2}\lambda K^{\frac{1}{2}}L^{\frac{1}{2}}$$
$$\frac{\partial \pounds}{\partial K} = r - \tfrac{1}{2}\lambda K^{-\frac{1}{2}}L^{\frac{3}{2}} = 0 \Longrightarrow r = \tfrac{1}{2}\lambda K^{-\frac{1}{2}}L^{\frac{3}{2}}$$
$$\text{These give: } \tfrac{w}{r} = 3\tfrac{K}{L}$$

   Now plug the given information (K=L and w=3) into this equation; $\frac{3}{r} = 3 * 1 \Longrightarrow r = 1$.

   Now given $w = 3$, $q = K^{\frac{1}{2}}L^{\frac{3}{2}}$, $r = 1$, *we can find critical points, which will be required in the following questions, as follows:*

$$\begin{array}{c} 3 = \frac{3}{2}\lambda Q/L \\ 1 = \frac{1}{2}\lambda Q/K \\ Q - L^{\frac{3}{2}}K^{\frac{1}{2}} = 0 \end{array} \quad \rightarrow \quad \begin{array}{c} L^* = K^* = \sqrt{Q} \\ 2/\sqrt{Q} = \lambda^* \end{array}$$

   (b) If the price of output is 1, and profits are 5 ($\pi = 5$) what is the level of output of the firm?

$$\pi = pq - wL - rK$$

   since it is given that;

$$\pi = 5,\ p = 1,\ w = 3,\ q = K^{\frac{1}{2}}L^{\frac{3}{2}},\ r = 1,\ K = L$$

   We have:

$$5 = q - 3L - L \text{ and } q = L^2$$

   then;

$$5 = L^2 - 4L \Longrightarrow L^2 - 4L - 5 = 0 \Longrightarrow L = 5 \text{ or } L = -1$$

   since $L > 0$, *we can conclude that* $L = 5$. *And then,* $5 = q - 4L \Longrightarrow q = 25$.

(c) Now solve for the profit function, being sure to check the second order conditions.

our profit function is

$$\pi(p,w,r) = \max_{L,K} pf(K,L) - wL - rK$$
$$\pi(p,w,r) = \max_{L,K} pK^{\frac{1}{2}}L^{\frac{3}{2}} - wL - rK$$

and this has the first order conditions:

$$p\frac{1}{2}K^{\frac{1}{2}-1}L^{\frac{3}{2}} - r = 0$$
$$p\frac{3}{2}K^{\frac{1}{2}}L^{\frac{3}{2}-1} - w = 0$$

solving the first one we get:

$$K = \frac{1}{4}p^2\frac{L^3}{r^2}$$

plugging this into the second one we get:

$$p\frac{3}{2}\left(\frac{1}{4}p^2\frac{L^3}{r^2}\right)^{\frac{1}{2}}L^{\frac{3}{2}-1} - w = 0$$
$$\frac{3}{4}p^2\frac{L^2}{r} - w = 0$$
$$L = \frac{2}{3}\frac{(3wr)^{\frac{1}{2}}}{p}$$

and

$$K = \frac{1}{4}p^2\frac{L^3}{r^2} = \frac{1}{4}p^2\frac{\left(\frac{2}{3}\frac{(3wr)^{\frac{1}{2}}}{p}\right)^3}{r^2} = \frac{2}{9p}\frac{3^{\frac{1}{2}}w^{\frac{3}{2}}}{r^{\frac{1}{2}}}$$

which means that our profit function is:

$$\pi(p,w,r) = p\left(\frac{2}{9p}\frac{3^{\frac{1}{2}}w^{\frac{3}{2}}}{r^{\frac{1}{2}}}\right)^{\frac{1}{2}}\left(\frac{2}{3}\frac{(3wr)^{\frac{1}{2}}}{p}\right)^{\frac{3}{2}} - w\left(\frac{2}{3}\frac{(3wr)^{\frac{1}{2}}}{p}\right) - r\frac{2}{9p}\frac{3^{\frac{1}{2}}w^{\frac{3}{2}}}{r^{\frac{1}{2}}}$$

which is not a pleasant sight. But heh, that's an answer... Except for one thing, we didn't check our second order conditions? There are three of these, write

$$\Pi(L,K) = pf(K,L) - wL - rK$$

then what we need for a maximum is:

$$\Pi_{LL} \leq 0$$
$$\Pi_{KK} \leq 0$$
$$\Pi_{LL}\Pi_{KK} - \Pi_{KL}^2 \geq 0$$

The first one is:

$$\Pi_{LL} = p\frac{\partial^2 f}{\partial L^2} = p\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)K^{\frac{1}{2}}L^{\frac{3}{2}-2} > 0$$

now this violates $\Pi_{LL} \leq 0$, so we know that what we found above is NOT the profit function. We have not found the top of the hill with regards to L, in fact we've found the bottom, if $\Pi_{LL} > 0$ then increasing L will only increase your profit. But let's check the other ones for completeness.

$$\Pi_{KK} = p\frac{\partial^2 f}{\partial K^2} = p\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)K^{\frac{1}{2}-2}L^{\frac{3}{2}} < 0$$
$$\Pi_{KL} = p\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)K^{\frac{1}{2}-1}L^{\frac{3}{2}-1}$$

$$\Pi_{LL}\Pi_{KK} - \Pi_{KL}^2 =$$
$$\left(p\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)K^{\frac{1}{2}}L^{\frac{3}{2}-2}\right)\left(p\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)K^{\frac{1}{2}-2}L^{\frac{3}{2}}\right) - \left(p\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)K^{\frac{1}{2}-1}L^{\frac{3}{2}-1}\right)^2 =$$
$$-\frac{3}{4}p^2\frac{L}{K} < 0$$

which is quite messy. And we have another violation. But forget that, wasn't this entire experience just way too painful? The point is that it is really hard to check all of the second order conditions for such a problem. It's hard to check a two by two matrix. Compare this with the situation where we do cost minimization above

5. Assume there are four firms in an industry all with the same total cost function $C(q) = 2q^2 + 8$. Type a firms have a fixed sunk cost of 8 type b firms have a fixed start up cost of 8.

   (a) Find the marginal cost, average variable cost, and supply curve of firms of type a.

$$
\begin{aligned}
MC &= 4q \\
AVC &= \frac{2q^2}{q} = 2q \\
MC &\geq AVC \\
4q &\geq 2q \\
q &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
P_s &= 4q \\
q &= \frac{1}{4} P_s
\end{aligned}
$$

$$
s(P_s) = \frac{1}{4} P_s
$$

(b) Find the marginal cost, average variable cost, and supply curve of firms of type b.

$$
\begin{aligned}
MC &= 4q \\
AVC &= \frac{2q^2 + 8}{q} \\
MC &\geq AVC \\
4q &\geq \frac{2q^2 + 8}{q} \\
q &\geq 2 \\
P_s &\geq 4(2) = 8
\end{aligned}
$$

$$
\begin{aligned}
P_s &= 4q \\
q &= \frac{1}{4} P_s
\end{aligned}
$$

$$
s(P_s) = \begin{cases} \frac{1}{4} P_s & P_s \geq 8 \\ 0 & P_s \leq 8 \end{cases}
$$

(c) Explain why the two types of firms have different levels of sunk costs when they are using the same technology.

These firms are at different points in their "life cycle." Type a firms have already made an irreversible investment in a plant, type b firms have not made that investment.