Three Examples of Exchange Economies

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1 Introduction

This handout will consider the three key examples of exchange economies. In all of the examples person 1 will have a Cobb-Douglas Utility function:

$$U_1(C_1, F_1) = C_1 F_1^{\alpha} \tag{1}$$

for $\alpha > 0$. Person 2 will either have a Cobb-Douglas, Leontief, or Linear (Perfect Substitutes) utility function:

$$\begin{aligned} U_2(C_2, F_2) &= C_2 F_2^{\beta}, & (2)\\ U_2(C_2, F_2) &= \min\{C_2, \beta F_2\}, & (3)\\ \text{or } U_2(C_2, F_2) &= C_2 + \beta F_2 & (4) \end{aligned}$$

for $\beta > 0$.

They will both begin with initial endowments $((C_{01}, F_{01}), (C_{02}, F_{02}))$ and the total amount of C and F in the economy is $C_{01} + C_{02} = C_0$, $F_{01} + F_{02} = F_0$.

We will find two things in each case. The contract curve—or the set of Pareto Efficient allocations—and the Walrasian or competitive equilibrium. The latter is a set of prices (p_c, p_f) such that both consumers maximize their utility and markets clear. We will assume that in this equilibrium $p_c > 0$ and $p_f > 0$ and that both parties always consume a strictly positive amount of both goods.

Notice that both are characterized the marginal rate of substitutions being equal for the two consumers, and the supply of both goods (C_0, F_0) being fully used.

2 Cobb-Douglas/Linear

While it might seem silly, it is easiest to work backwards. The hardest one is the first one. Thus:

$$\begin{aligned} U_1(C_1, F_1) &= C_1 F_1^{\alpha}\\ U_2(C_2, F_2) &= C_2 + \beta F_2 \end{aligned}$$

$$\begin{aligned} MRS_1 &= \frac{MU_c^1}{MU_f^1} = \frac{\frac{U_1}{C_1}}{\alpha\frac{U_1}{F_1}} = \frac{F_1}{\alpha C_1}\\ MRS_2 &= \frac{1}{\beta} \end{aligned}$$

$$MRS_1 = MRS_2$$

$$\begin{aligned} \frac{F_1}{\alpha C_1} &= \frac{1}{\beta} \\ F_1 &= \frac{\alpha}{\beta} C_1 \end{aligned}$$

2.1 The Contract Curve:

Now we have three conditions:

$$\begin{aligned} F_1 &= \frac{\alpha}{\beta} C_1 \quad (5) \\ F_1 + F_2 &= F_0 \\ C_1 + C_2 &= C_0 \end{aligned}$$

but the first one is a complete description of the contact curve. I would like to mention that since I will always assume both parties consumption of both goods is strictly positive that this requires that:

$$F_0 = \frac{\alpha}{\beta} C_0$$

so that we meet this condition. Do not worry, this will be part of the set up of the problem.

2.2 The Walrasian Equilibrium

In this case we have:

$$MRS_1 = \frac{MU_c^1}{MU_f^1} = \frac{p_c}{p_f} = MRS_2$$

but since $MRS_2 = \frac{1}{\beta}$ we know that $\frac{p_c}{p_f} = \frac{1}{\beta}$. Let's just set $p_f = \beta$, $p_c = 1$. Now unfortunately to find the final allocation we have to maximize person 1's utility where their income is how much they will earn on their bundle of goods $I_1 = p_c C_{01} + p_f F_{01}$.

$$\frac{F_1}{\alpha C_1} = \frac{p_c}{p_f}$$

$$p_c C_1 + p_f F_1 = p_c C_{01} + p_f F_{01}$$

substituting in $p_f = \beta$, $p_c = 1$ we get:

$$\frac{F_1}{\alpha C_1} = \frac{1}{\beta} \Rightarrow F_1 = \frac{\alpha}{\beta} C_1$$

$$C_1 + \beta F_1 = C_{01} + \beta F_{01}$$

$$\begin{aligned} C_1 + \beta \left(\frac{\alpha}{\beta} C_1 \right) &= C_{01} + \beta F_{01} \\ C_1 (\alpha + 1) &= C_{01} + \beta F_{01} \end{aligned}$$

$$C_1 = \frac{1}{\alpha + 1}(\beta C_{01} + F_{01}) \tag{6}$$

$$F_1 = \frac{\alpha}{\beta}C_1 = \frac{\alpha}{\beta}\frac{1}{\alpha+1}(\beta C_{01} + F_{01}) = \frac{\alpha}{1+\alpha}\frac{1}{\beta}(\beta C_{01} + F_{01}) \tag{7}$$

and we are done.

3 Cobb-Douglas/Leontief

For person 2 we do not have a Marginal Rate of Substitution, rather we know that:

$$C_2 = \beta F_2$$

but that does not make things that difficult.

3.1 Contract Curve

We have three conditions:

$$\begin{aligned} C_2 &= \beta F_2 \\ F_1 + F_2 &= F_0 \\ C_1 + C_2 &= C_0 \end{aligned}$$

And we notice that since consumptions must always be strictly positive this means that:

$$C_0 = \beta F_0. \tag{8}$$

In this case we will actually use this. Regardless we generally want to write the contract curve as F_1 as a function of C_1, but this is almost trivial to do:

$$\begin{aligned} C_2 &= C_0 - C_1 \\ F_2 &= F_0 - F_1 \end{aligned}$$

$$C_0 - C_1 = \beta(F_0 - F_1)$$

$$\begin{aligned} \beta F_1 &= \beta F_0 - C_0 + C_1 \\ F_1 &= F_0 - \frac{1}{\beta}C_0 + \frac{1}{\beta}C_1 \end{aligned} \tag{9}$$

Notice that since $C_0 = \beta F_0$ this simplifies to:

$$F_1 = F_0 - \frac{1}{\beta}(\beta F_0) + \frac{1}{\beta}C_1 = \frac{1}{\beta}C_1 \tag{10}$$

3.2 The Walrasian Equilibrium

In this case we will use the contract curve to find the equilibrium prices, and then use that to find the final allocations. We have:

$$\frac{F_1}{\alpha C_1} = \frac{p_c}{p_f}$$

and $F_1 = \frac{1}{\beta}C_1$ so this gives us:

$$\begin{aligned} \frac{\left(\frac{1}{\beta}C_1\right)}{\alpha C_1} &= \frac{p_c}{p_f} \\ \frac{1}{\alpha\beta} &= \frac{p_c}{p_f} \end{aligned} \tag{11}$$

Again it is convenient to set $p_f = \alpha\beta$, $p_c = 1$ and

$$\begin{aligned} \frac{F_1}{\alpha C_1} &= \frac{1}{\alpha\beta} \\ F_1 &= \frac{1}{\beta}C_1 \end{aligned}$$

$$\begin{aligned} C_1 + \alpha\beta F_1 &= C_{01} + \alpha\beta F_{01} \\ C_1 + \alpha\beta\left(\frac{1}{\beta}C_1\right) &= C_{01} + \alpha\beta F_{01} \\ C_1(\alpha + 1) &= C_{01} + \alpha\beta F_{01} \\ C_1 &= \frac{1}{1+\alpha}\left(C_{01} + \alpha\beta F_{01}\right) \end{aligned} \tag{12}$$

$$F_1 = \frac{1}{\beta}C_1 = \frac{1}{\beta}\frac{1}{1+\alpha}\left(C_{01} + \alpha\beta F_{01}\right) \tag{13}$$

And we are done.

4 Cobb-Douglas/Cobb-Douglas

This is the most complicated one because:

$$MRS_2 = \frac{\frac{U_2}{C_2}}{\beta\frac{U_2}{F_2}} = \frac{F_2}{\beta C_2}$$

And so we have no "cheap trick" to find either the prices or the contract curve.

4.1 The Contract Curve

All we have is

$$MRS_1 = \frac{F_1}{\alpha C_1} = \frac{F_2}{\beta C_2} = MRS_2$$

$$\begin{aligned} F_1 + F_2 &= F_0 \\ C_1 + C_2 &= C_0 \end{aligned}$$

So we have to substitute out for (C_2, F_2) $C_2 = C_0 - C_1$ $F_2 = F_0 - F_1$

$$\frac{F_1}{\alpha C_1} = \frac{F_0 - F_1}{\beta (C_0 - C_1)}$$

$$\begin{aligned} F_1 \beta (C_0 - C_1) &= \alpha C_1 (F_0 - F_1) \\ F_1 \beta (C_0 - C_1) &= \alpha C_1 F_0 - \alpha C_1 F_1 \\ F_1 \beta (C_0 - C_1) + \alpha C_1 F_1 &= \alpha C_1 F_0 \\ F_1 ((\alpha - \beta) C_1 + \beta C_0) &= \alpha C_1 F_0 \\ F_1 &= \frac{\alpha C_1 F_0}{\beta C_0 + (\alpha - \beta) C_1} \qquad (14) \end{aligned}$$

Which is not that difficult but for the first time it is not a simple line. It is instead an increasing and either concave or convex function.

4.2 The Walrasian Equilibrium

Like with the contract curve their is no cheap trick. Thanks to Walras's law we only have to clear one market and we can set one price equal to one. Let us delay the decision on which price is one for now.

$$\begin{aligned} \frac{F_1}{\alpha C_1} &= \frac{p_c}{p_f} \Rightarrow F_1 = \alpha C_1 \frac{p_c}{p_f} \\ p_c C_1 + p_f F_1 &= p_c C_{01} + p_f F_{01} \end{aligned}$$

$$\begin{aligned} p_c C_1 + p_f \left(\alpha C_1 \frac{p_c}{p_f} \right) &= p_c C_{01} + p_f F_{01} \\ C_1 p_c (\alpha + 1) &= p_c C_{01} + p_f F_{01} \\ C_1 &= \frac{1}{\alpha + 1} \frac{1}{p_c} (p_c C_{01} + p_f F_{01}) \end{aligned}$$

I do the exact same thing again for person 2:

$$\begin{aligned} \frac{F_2}{\beta C_2} &= \frac{p_c}{p_f} \Rightarrow F_2 = \beta C_2 \frac{p_c}{p_f} \\ p_c C_2 + p_f F_2 &= p_c C_{02} + p_f F_{02} \end{aligned}$$

$$
\begin{aligned}
p_c C_2 + p_f \left(\beta C_2 \frac{p_c}{p_f} \right) &= p_c C_{02} + p_f F_{02} \\
C_2 p_c \left(\beta + 1 \right) &= p_c C_{02} + p_f F_{02} \\
C_2 &= \frac{1}{\beta + 1} \frac{1}{p_c} \left(p_c C_{02} + p_f F_{02} \right)
\end{aligned}
$$

And I want:

$$
\begin{aligned}
C_1 \left(p_c, p_f \right) + C_2 \left(p_c, p_f \right) &= C_0 \\
\frac{1}{\alpha + 1} \frac{1}{p_c} \left(p_c C_{01} + p_f F_{01} \right) + \frac{1}{\beta + 1} \frac{1}{p_c} \left(p_c C_{02} + p_f F_{02} \right) &= C_0
\end{aligned}
$$

Now the choice of setting p_c or p_f to one is arbitrary, but looking at this market clearing equation I find it a lot easier just to set $p_c = 1$. Then my problem is:

$$
\begin{aligned}
\frac{1}{\alpha + 1} \left(C_{01} + p_f F_{01} \right) + \frac{1}{\beta + 1} \left(C_{02} + p_f F_{02} \right) &= C_0 \\
\frac{1}{\alpha + 1} C_{01} + \frac{1}{\alpha + 1} p_f F_{01} + \frac{1}{\beta + 1} C_{02} + \frac{1}{\beta + 1} p_f F_{02} &= C_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\alpha + 1} p_f F_{01} + \frac{1}{\beta + 1} p_f F_{02} &= C_0 - \frac{1}{\alpha + 1} C_{01} - \frac{1}{\beta + 1} C_{02} \\
\left(\frac{1}{\alpha + 1} F_{01} + \frac{1}{\beta + 1} F_{02} \right) p_f &= C_0 - \frac{1}{\alpha + 1} C_{01} - \frac{1}{\beta + 1} C_{02}
\end{aligned}
$$

$$
p_f = \frac{C_0 - \frac{1}{\alpha+1} C_{01} - \frac{1}{\beta+1} C_{02}}{\frac{1}{\alpha+1} F_{01} + \frac{1}{\beta+1} F_{02}} \tag{15}
$$

and of course I promise you that in any problem I assign this will be a simple fraction, as in $p_f = x/y$ where x and y are natural numbers—generally either x or y will be one. Then to finish the description of the equilibrium we note that:

$$
\left(C_1, F_1 \right) = \left(\frac{1}{\alpha + 1} \left(C_{01} + p_f F_{01} \right), \frac{1}{p_f} \frac{\alpha}{\alpha + 1} \left(C_{01} + p_f F_{01} \right) \right) \tag{16}
$$

and we are done.