Corner Solutions–
when people don't buy everything

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1 A reality check

In mathematics, a *corner solution* is when one or more variables in an optimization is constrained at either the maximum or minimum value it can take. We are most interested in demand being non-negative, i.e. you can not buy a negative amount of a good. If a consumer would "want" to buy a negative amount, we constrain their demand at zero—in other words they simply do not consume any of that good.

In reality I am pretty sure that the most obvious fact about consumers is that they don't buy all goods. If you asked the manager of a large grocery store she or he could probably tell you how many people–of what type–buy a certain good. It would be obvious to him or her that not every customer buys every good. Forget "brand loyalty" their are some people who don't use certain cleaning supplies, etcetera. Do you buy mouthwash? Many don't. What is your order at your favorite coffee shop? I bet you have one coffee that you *always* order (maybe two or three). This doesn't mean you despise the rest of the menu, its just this is your favorite. Even more, do you have a bucket list? That is, by definition, a list of things you would like to do but either don't have the time or money to do. If you consume everything, how can you have a bucket list? Personally one thing I am vaguely interested in but would never do is rent a race car to drive around a racetrack in. I have heard this costs about 8K USD for five minutes, and there's no way that is worth it.

It isn't a discreteness problem. You could, for example, rent any fancy sports car you want to for a day... might be worth it for you but not for me. It isn't an effect of me not wanting some goods. If someone gave me a present of either a sports car for a day or driving a race car I would be very excited. Besides myself with joy. It is just that the amount of utility I would gain per Lira (or Dollar) spent is just not high enough. To use the lingo, the bang for the buck is too low.

So why don't we analyze this when we do Utility maximization? The answer is simply complexity. If we consider corner solutions then we have to go through a case based analysis—what if $x = 0$, what if $y = 0$, what if $x = 0$ and $y = 0$—and that is hard, so we focus on the simplest case, where the solution is *interior* or $x > 0$ and $y > 0$. If the solution is interior then we know for a fact that the first order conditions must be satisfied, and if the objective function has convex indifference curves we can be sure that the second order conditions will be satisfied. It is nothing more than that, corner solutions are hard.

So it is important to know that corner solutions exist, it is also important to know how to find them, but in the end we won't be making you solve any problems with corner solutions, except for quasi-linear or linear utility (perfect substitutes). Linear utility is a hard case to analyze simply because for most prices the solution is always a corner solution.

2 Complimentary Slackness—the formal mathematical technique.

For constrained maximization the methodology we should be using is the Kuhn-Tucker methodology. Why don't we teach this technique? We do in the Math for ECON class, but the general answer is because for the utility functions we analyze (CES and Quasi-linear) most of the time Lagrange is enough. To be precise as long as our preferences satisfy *local non-satiation* we are fine, local non-satiation is automatically satisfied if our preferences are *monotonic* or *strongly monotonic* (the weak and strong version of "more is better").

Except... of course... if we end up at a corner solution. Ergh, then we need the full strength of Kuhn-Tucker, namely *complimentary slackness.*

Definition 1 *Complimentary slackness: at a maximum of the function $L(x, y, ...)$ we must have:*

$$\frac{\partial L}{\partial x} \leq 0, \quad x \geq 0, \quad \frac{\partial L}{\partial x} x = 0$$

This of course allows for the possibility that $x = 0$ and $\frac{\partial L}{\partial x} < 0$, alternatively the restriction on x could be slack and then $x > 0$ and $\frac{\partial L}{\partial x} = 0$. So one of the two inequalities can be "slack" or not satisfied with equality: there are complimentary inequality conditions.

Let me show a few examples.

Example 2 *Maximizing consumer surplus.*

This is the standard method underlying the classic $Q = a - bP$ that we love so much. I am certain you have never seen it generated this way before, but you will need to understand this for ECON 204 and beyond. We can think of a consumer as maximizing:

$$CS(Q) = B(Q) - PQ$$

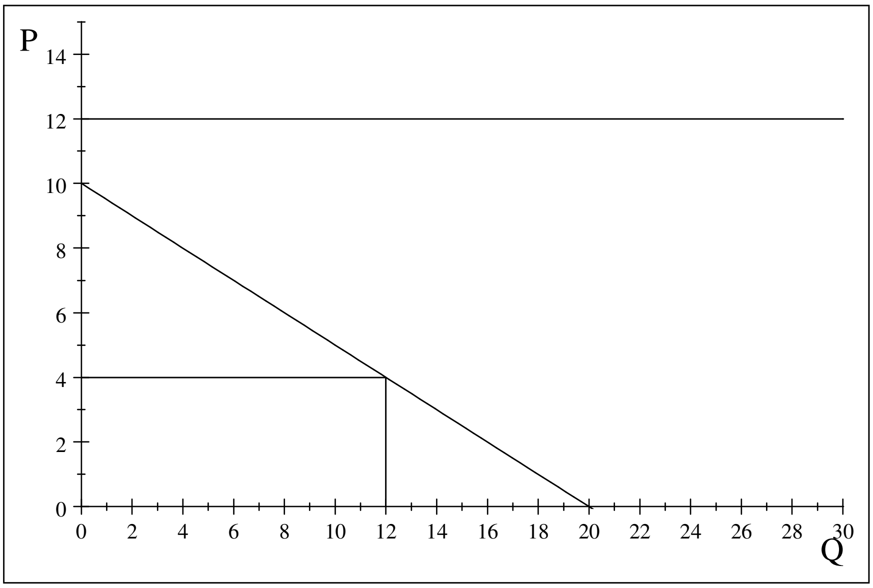
or consumer surplus.[1] *The first derivative of this function is:*

$$\frac{dCS}{dQ} = \frac{dB}{dQ} - P = MB - P,$$

and the second derivative is $\frac{d^2CS}{dQ^2} = \frac{d^2B}{dQ^2}$ and we simply assume that $B(Q)$ is concave so we don't need to worry about that (we assume $\frac{d^2B}{dQ^2} < 0$). So the solution satisfies the first derivative is equal to zero, right? Wrong, as I am

[1] Notice that in order to do this properly they must have quasi-linear utility (see below) and a sufficiently large income.

sure you know if the price is too high this customer won't buy anything. Let me assume that $B(Q) = \left(\frac{a}{b} - \frac{1}{2b}Q\right)Q$ *or* $MB = \frac{a}{b} - \frac{1}{b}Q$ *which would give us our standard linear demand curve. In the graph below I set* $a = 20$, $b = 2$.



at a nice normal price like 4, we trace a line over to the demand curve ($Q = a - bP$) *and find that the quantity they want to buy is 12. On the other hand at some crazy price like 12 when we trace a horizontal line it never touches the demand curve, which means the demand is zero.*

Now let's look at the full complimentary slackness conditions when $MB = \frac{a}{b} - \frac{1}{b}Q$ *they are:*

$$\begin{aligned} \frac{dCS}{dQ} &= \frac{a}{b} - \frac{1}{b}Q - P \leq 0 \\ Q &\geq 0 \\ \frac{dCS}{dQ}Q &= 0 \end{aligned}$$

we can clearly see that the highest $\frac{dCS}{dQ}$ *can be is* $\frac{a}{b} - P$*, so if* $P > \frac{a}{b}$ *then* $\frac{dCS}{dQ} < 0$. *That is fine, because then complimentary slackness tells us that we must have* $Q = 0$*, which is the conclusion we expect.*

Example 3 *Quasi-linear utility:* $u(x, y) = \frac{x^{1-\sigma}}{1-\sigma} + y$ *for* $\sigma > 0$ *and* $\sigma \neq 1$. *In this case we need to set up a proper objective function, if you don't mind I would like to set the price of* y (p_y) *to one. Then this objective function is:*

$$L_{ql}(x, y, \lambda) = \frac{x^{1-\sigma}}{1-\sigma} + y - \lambda(p_x x + y - I)$$

and the first derivatives are:

$$\begin{aligned}\frac{\partial L_{ql}}{\partial x} &= \frac{1}{x^{\sigma}} - \lambda p_x \\ \frac{\partial L_{ql}}{\partial y} &= 1 - \lambda \\ \frac{\partial L_{ql}}{\partial \lambda} &= -\left(p_x x + y - I\right)\end{aligned}$$

now this utility function is strictly monotonic, so in any maximum we must have $\frac{\partial L_{ql}}{\partial \lambda} = 0$*. Further we notice that for small enough* x $\frac{1}{x^{\sigma}}$ *can be arbitrarily large, so we must have* $x > 0$ *and* $\frac{\partial L_{ql}}{\partial x} = 0$*. Of course we can not be sure that* $\frac{\partial L_{ql}}{\partial y} = 0$*, we might have* $\lambda > 1$*. From the first order condition* $\frac{\partial L_{ql}}{\partial x} = 0$ *we know that* $\lambda = \frac{1}{p_x x^{\sigma}}$*, and if* $\lambda > 1$ *then we must have this being strictly greater than one when* $x = \frac{I}{p_x}$ *or we must have:*

$$\begin{aligned}\frac{1}{p_x\left(\frac{I}{p_x}\right)^{\sigma}} &> 1 \\ \frac{1}{p_x^{1-\sigma} I^{\sigma}} &> 1\end{aligned}$$

this boils down to "income being pretty low." The impact of the price of x *on this condition depends on* σ*. If* $\sigma > 1$ *then a high* p_x *makes this condition harder to satisfy, if* $\sigma < 1$ *a higher* p_x *makes this condition easier to satisfy.*

***An Easier Way**:*

No no, I wouldn't dream of leaving it there. There is a much simpler way to proceed in this case. Assume the solution is interior*, or to be specific that* $y^* > 0$*. See? I told you we were analyzing corner solutions and then I tell you to assume them away. Do you see why we don't spend too much time on them? So if we assume that we have an interior solution then* $\lambda = 1$ *and:*

$$\begin{aligned}\frac{\partial L_{ql}}{\partial x} &= \frac{1}{x^{\sigma}} - p_x = 0 \\ x &= \left(\frac{1}{p_x}\right)^{\frac{1}{\sigma}} = x\left(p_x\right)\end{aligned}$$

therefore:

$$\begin{aligned}p_x\left(\frac{1}{p_x}\right)^{\frac{1}{\sigma}} + y &= I \\ y &= I - p_x^{\frac{1}{\sigma}(\sigma-1)} = y\left(p_x, I\right)\end{aligned}$$

Now, when will we have a corner solution? When $y\left(p_x, I\right) \leq 0$ *or...*

$$I - p_x^{\frac{1}{\sigma}(\sigma-1)} \leq 0$$

which is the same condition as above, after a little bit of manipulation. However this was a much easier way to proceed.

Example 4 *Linear utility,* $u(x,y) = \alpha x + \beta y$

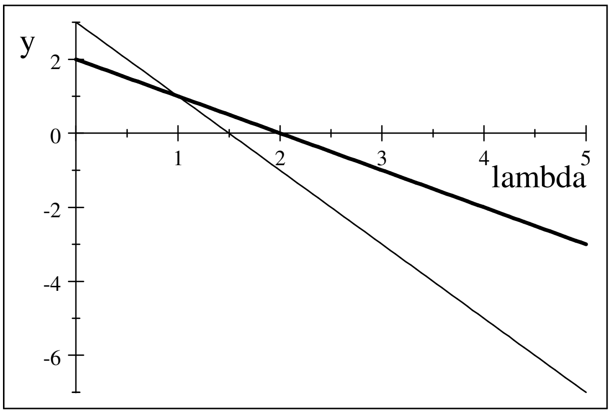
$$\begin{aligned} L_{qllin}(x,y,\lambda) &= \alpha x + \beta y - \lambda(p_x x + p_y y - I) \\ &= (\alpha - \lambda p_x)x + (\beta - \lambda p_y)y + \lambda I \end{aligned}$$

proceeding formally we look at the first derivatives:

$$\begin{aligned} \frac{\partial L_{lin}}{\partial x} &= \alpha - \lambda p_x \\ \frac{\partial L_{lin}}{\partial y} &= \beta - \lambda p_y \\ \frac{\partial L_{lin}}{\partial \lambda} &= -(p_x x + p_y y - I) \end{aligned}$$

like before this utility function is strictly monotonic so we know $\frac{\partial L_{lin}}{\partial \lambda} = 0$*, but what do we do about the other two conditions? These are, in short, a function of* λ *so if we choose* λ *high enough both will be negative, and if we choose* λ *too low then both will be positive. So how do we choose* λ*?*

Well we need to go back to the complimentary slackness conditions. It says that if $x > 0$ *we must have* $\frac{\partial L_{lin}}{\partial x} = 0$ *and if* $x = 0$ *we must have* $\frac{\partial L_{lin}}{\partial x} \leq 0$*. The reasoning for this is transparent enough, if* $\frac{\partial L_{lin}}{\partial x} > 0$ *that immediately tells you that you should increase* x*, so you better not have this if you are trying to claim this is a maximizing strategy. This tells us we have to choose* λ *such that* $\frac{\partial L_{lin}}{\partial x} \leq 0$ *and* $\frac{\partial L_{lin}}{\partial y} \leq 0$*. But how about choosing it so high that* $\frac{\partial L_{lin}}{\partial x} < 0$ *and* $\frac{\partial L_{lin}}{\partial y} < 0$*? No, that won't work because then* $x = y = 0$ *and that can't be utility maximizing (this function is strictly monotonic). So we have to have* $\max\left(\frac{\partial L_{lin}}{\partial x}, \frac{\partial L_{lin}}{\partial y}\right) = 0$*. Good. But does that help? In the graph below I graph* $\frac{\partial L_{lin}}{\partial x}$ *and* $\frac{\partial L_{lin}}{\partial y}$ *as a function of* λ*, when* $\alpha = 2$*,* $p_x = 1$ *and* $\beta = 3$*,* $p_y = 2$*.*



In this graph the thick line is $\frac{\partial L_{lin}}{\partial x}$ *and the thin one is* $\frac{\partial L_{lin}}{\partial y}$*. Notice that while* $\frac{\partial L_{lin}}{\partial y}$ starts out *higher by the time it comes to choosing the critical* λ *(when*

$\frac{\partial L_{lin}}{\partial x} \leq 0$ and $\frac{\partial L_{lin}}{\partial y} \leq 0$) it is lower, and that the right choice for λ is 2, when $\frac{\partial L_{lin}}{\partial x} = 0 > \frac{\partial L_{lin}}{\partial y}$. This immediately tells us that $y = 0$ and thus $x = \frac{I}{p_x}$.

But surely their must be a way to proceed that is a little easier, no? Yes there is look at the expressions:

$$\begin{aligned} \frac{\partial L_{lin}}{\partial x}\frac{1}{p_x} &= \frac{\alpha}{p_x} - \lambda \\ \frac{\partial L_{lin}}{\partial y}\frac{1}{p_y} &= \frac{\beta}{p_y} - \lambda \end{aligned}$$

obviously if $\frac{\partial L_{lin}}{\partial x} \leq 0$ then $\frac{\partial L_{lin}}{\partial x}\frac{1}{p_x} \leq 0$ and vice-versa. However now we can easily figure out the optimal λ to set one of them equal to zero, and $\lambda = \frac{\alpha}{p_x}$ will work when

$$\frac{\partial L_{lin}}{\partial y}\frac{1}{p_y} = \frac{\beta}{p_y} - \frac{\alpha}{p_x} < 0$$

or

$$\frac{\beta}{p_y} < \frac{\alpha}{p_x}$$

And that, you probably recall, is the Bang for the Buck condition—the marginal utility per unit of money.

$$BfB_x = \frac{MU_x}{p_x} \ .$$

So, if $BfB_x = \frac{\alpha}{p_x} > \frac{\beta}{p_y} = BfB_y$ then $x = \frac{I}{p_x}$ and $y = 0$. The reverse is obviously true if $\frac{\beta}{p_y} > \frac{\alpha}{p_x}$. and in the knife edge case where $\frac{\beta}{p_y} = \frac{\alpha}{p_x}$ the consumer doesn't care what they buy as long as they spend all their income.

3 An intuitive approach using Bang for the Bucks

A more intuitive approach is to use the Bang for the Buck from the start (BfB). The bang for the buck of good x is:

$$BfB_x = \frac{MU_x}{p_x} \ .$$

It is the amount of happiness you get *per lira* (or whatever your favorite unit of money is). The power of this technique is that it is ordinal[2] and it doesn't

[2] To see this notice that $\tilde{u}(x,y) = f(u(x,y))$ then $\frac{\partial \tilde{u}}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$ and $\frac{\partial \tilde{u}}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}$, so if

$$\widetilde{BfB}_x = \frac{\partial \tilde{u}}{\partial x}\frac{1}{p_x} \geq \frac{\partial \tilde{u}}{\partial y}\frac{1}{p_y} = \widetilde{BfB}_y$$

then this means:

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\frac{1}{p_x} \geq \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}\frac{1}{p_y}$$

and since $\frac{\partial f}{\partial u} > 0$ this equivalent to

$$BfB_x = \frac{\partial u}{\partial x}\frac{1}{p_x} \geq \frac{\partial u}{\partial y}\frac{1}{p_y} = BfB_y \ .$$

depend on any other goods, it is a stand alone concept. It can also be used as a guide for maximization:

Proposition 5 *If preferences are convex and $BfB_x > BfB_y$ then you should increase x and decrease y.*

As someone who spends way too much time characterizing equilibrium, you probably don't see the point. I mean, who cares about what to do when you are *outside* of the optimum. I apologize for training you so badly. You have spent so much time trying to learn how to find optima that you never thought about the basic insight *underpinning* this optimum. This will now come back to haunt you.

Why? Because of the obvious corollary:

Corollary 6 *If when $y = 0$ there is an $x^* > 0$ and $z^* > 0$ such that*

$$BfB_y < BfB_x = BfB_z$$

then the optimal consumption of y is zero.

I only mentioned a third good there so that you don't get hung up thinking this only holds for two goods. It holds for any number of goods, if goods are consumed in a strictly positive amount, then their bang for the bucks must be the same. If their consumption is zero, they have a lower bang for the buck. Of course I have already been using this technique. It is possible to analyze these problems without using it but it is awfully hard. For example when I re-scaled the first derivatives of the linear utility function:

$$\begin{aligned} \frac{\partial L_{lin}}{\partial x}\frac{1}{p_x} &= \frac{\alpha}{p_x} - \lambda = BfB_x - \lambda \\ \frac{\partial L_{lin}}{\partial y}\frac{1}{p_y} &= \frac{\beta}{p_y} - \lambda = BfB_y - \lambda \end{aligned}$$

and when $BfB_x = \frac{\alpha}{p_x} > \frac{\beta}{p_y} = BfB_y$ we conclude $y^* = 0$—you don't consume any y. The same conclusion as above, but relying directly on the bang for the bucks without going through all that mathematical setup. I can also do the same thing for the quasi-linear case:

$$\begin{aligned} \frac{\partial L_{ql}}{\partial x}\frac{1}{p_x} &= \frac{1}{p_x x^\sigma} - \lambda \\ \frac{\partial L_{ql}}{\partial y}\frac{1}{p_y} &= 1 - \lambda \ , \end{aligned}$$

(remember that I set $p_y = 1$). And like argued above, if $\frac{1}{p_x(x^*)^\sigma} > 1$ then $y^* = 0$. Of course in this case we have $x^* = \frac{I}{p_x}$, but that is not important. Let me do one final example, simply to show you how it would go if we had more than two goods. Its a pain in the neck, but I really just want to show you the method, and most importantly show you an example where two goods are consumed in a strictly positive amount.

Example 7 *A three good quasi-linear utility function:*

$$u(x,y,m) = x^{\alpha}y^{\beta} + m$$

in this case we assume $\alpha > 0$, $\beta > 0$ *and* $\alpha + \beta < 1$. *This makes the function strictly concave. We will need our objective function and derivatives later, so the objective function is:*

$$L_{3g}(x,y,m,\lambda) = x^{\alpha}y^{\beta} + m - \lambda(p_x x + p_y y + p_m m - I)$$

and the derivatives are:

$$\begin{aligned}
\frac{\partial L_{3g}}{\partial x} &= \alpha\frac{x^{\alpha}y^{\beta}}{x} - \lambda p_x \\
\frac{\partial L_{3g}}{\partial y} &= \beta\frac{x^{\alpha}y^{\beta}}{y} - \lambda p_y \\
\frac{\partial L_{3g}}{\partial m} &= 1 - \lambda p_m \\
\frac{\partial L_{3g}}{\partial \lambda} &= -(p_x x + p_y y + p_m m - I)
\end{aligned}$$

However instead of working with these for now, lets look at the bang for the bucks:

$$\begin{aligned}
BfB_x &= \alpha\frac{x^{\alpha}y^{\beta}}{p_x x} = \frac{\alpha}{p_x}\frac{y^{\beta}}{x^{1-\alpha}} \\
BfB_y &= \beta\frac{x^{\alpha}y^{\beta}}{p_y y} = \frac{\beta}{p_y}\frac{x^{\alpha}}{y^{1-\beta}} \\
BfB_m &= \frac{1}{p_m}\ .
\end{aligned}$$

Let's just be lazy and let $p_m = 1$ *like before. From the second formulation of the bang for the bucks you can see that if either* $x \to 0$ *or* $y \to 0$ *then* $BfB_x \to \infty$ *or* $BfB_y \to \infty$, *thus we know that* $x^* > 0$ *and* $y^* > 0$ *or:*

$$\begin{aligned}
\alpha\frac{x^{\alpha}y^{\beta}}{p_x x} &= \beta\frac{x^{\alpha}y^{\beta}}{p_y y} \\
\alpha p_y y &= \beta p_x x \\
x &= y\frac{\alpha}{\beta}\frac{p_y}{p_x}
\end{aligned}$$

Now things get a little complicated, if $m^* > 0$ *then we know that* $\lambda = 1$ *and:*

$$\frac{\partial L_{3g}}{\partial y} = \beta\frac{x^{\alpha}y^{\beta}}{y} - p_y = 0\ .$$

Using what we just found out about the optimal ratio of x and y this means:

$$\begin{aligned}
\beta\frac{\left(y\frac{\alpha}{\beta}\frac{p_y}{p_x}\right)^{\alpha}y^{\beta}}{y}-p_y &= 0\\
\beta\left(\frac{\alpha}{\beta}\frac{p_y}{p_x}\right)^{\alpha} &= p_y y^{1-\alpha-\beta}\\
\beta^{1}\left(\beta^{-\alpha}\right)\alpha^{\alpha}\left(p_y\right)^{\alpha}p_x^{-\alpha}\left(p_y\right)^{-1} &= y^{1-\alpha-\beta}\\
\beta^{1}\left(\beta^{-\alpha}\right)\alpha^{\alpha}\left(p_y\right)^{\alpha}p_x^{-\alpha}\left(p_y\right)^{-1} &= y^{1-\alpha-\beta}\\
\frac{\alpha^{\alpha}\beta^{1-\alpha}}{p_x^{\alpha}p_y^{1-\alpha}} &= y^{1-\alpha-\beta}\\
y &= \left(\frac{\alpha^{\alpha}\beta^{1-\alpha}}{p_x^{\alpha}p_y^{1-\alpha}}\right)^{\frac{1}{1-\alpha-\beta}}
\end{aligned}$$

$$\begin{aligned}
x &= \left(\frac{\alpha^{\alpha}\beta^{1-\alpha}}{p_x^{\alpha}p_y^{1-\alpha}}\right)^{\frac{1}{1-\alpha-\beta}}\frac{\alpha}{\beta}\frac{p_y}{p_x}\\
x^{1-\alpha-\beta} &= \frac{\alpha^{\alpha}\beta^{1-\alpha}}{p_x^{\alpha}p_y^{1-\alpha}}\frac{\alpha^{1-\alpha-\beta}}{\beta^{1-\alpha-\beta}}\frac{\left(p_y\right)^{1-\alpha-\beta}}{\left(p_x\right)^{1-\alpha-\beta}}\\
x &= \left(\frac{\alpha^{1-\beta}\beta^{\beta}}{p_y^{\beta}p_x^{1-\beta}}\right)^{\frac{1}{1-\alpha-\beta}}
\end{aligned}$$

and then from the budget constraint we know that:

$$\begin{aligned}
p_x x+p_y y+m &= I\\
p_x\left(\frac{\alpha^{1-\beta}\beta^{\beta}}{p_y^{\beta}p_x^{1-\beta}}\right)^{\frac{1}{1-\alpha-\beta}}+p_y\left(\frac{\alpha^{\alpha}\beta^{1-\alpha}}{p_x^{\alpha}p_y^{1-\alpha}}\right)^{\frac{1}{1-\alpha-\beta}}+m &= I
\end{aligned}$$

$$m=I-p_x\left(\frac{\alpha^{1-\beta}\beta^{\beta}}{p_y^{\beta}p_x^{1-\beta}}\right)^{\frac{1}{1-\alpha-\beta}}-p_y\left(\frac{\alpha^{\alpha}\beta^{1-\alpha}}{p_x^{\alpha}p_y^{1-\alpha}}\right)^{\frac{1}{1-\alpha-\beta}}.$$

On the other hand, if we don't consume the warm glow of money the problem is much simpler. Like before we have $x=y\frac{\alpha}{\beta}\frac{p_y}{p_x}$ and now budget balancing is:

$$\begin{aligned}
p_x x+p_y y+0 &= I\\
p_x\left(y\frac{\alpha}{\beta}\frac{p_y}{p_x}\right)+p_y y+0 &= I\\
p_y y\frac{\alpha+\beta}{\beta} &= I\\
y &= \frac{\beta}{\alpha+\beta}\frac{I}{p_y}
\end{aligned}$$

and of course $x=\frac{\alpha}{\alpha+\beta}\frac{I}{p_x}$. I.e. standard Cobb-Douglas demand curves.