A Critique of Continuous Models in Game Theory: The Bertrand and the Hotelling

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Abstract

This paper critiques the use of continuous models when this usage results in the best response not being well defined. It shows how in the Bertrand and Hotelling model the simpler discrete model results in clearer best responses and transparent results.

Interesting enough these results also strengthen our understanding. While it is not a novel insight, in the Bertrand game with asymmetric marginal costs we have equilibria that do not require changing the sharing rule. In the Hotelling model it shows that the equilibrium(ia) are actually the unique strategies to survive iterated deletion of dominated strategies.

1 Introduction

It is an interesting question whether the world is actually finite or continuous. I understand physicists are currently investigating the possibility that both might be true, but for most of us the choice is really a matter of convenience. After all—as Gödel (1931) showed—mathematics cannot even be internally perfect, it is simply a useful descriptive tool. This is well understood in Economics. For example in competitive analysis we assume the continuum for convenience. I have never been to a market that would sell me π apples, and a price of π would be—if not illegal—problematic. However even though our model might predict this result, we are not concerned. The model is not reality; it is a tool for understanding.

In Game Theory the preference should be the reverse. As Nash (1950) elegantly explained, if there are a finite number of options then best responses are well defined and all one needs to do is intersect them to find the equilibria. Of course in many game theory models (for example Cournot, 1838) the best response is always well defined, but if it does cause a problem we suggest the analyst should prefer a large finite number of strategies.

The two examples in this paper where the best response fails to exist (the Bertrand and Hotelling models) share the characteristic that one wants to be "close to" but not "the same" as the others. This causes an open set problem and the non-existence of best responses. In both cases equilibria do exist, but they magically pop into existence at the only place best responses are defined. Despite this fact the textbook model in both cases is the continuum.

Interestingly enough, we also derive stronger and clearer results. In the Bertrand game if we assume all variables are in (very small) units we can easily prove Bertrand's famous undercutting insight. Furthermore when the firm's marginal costs are not the same we can derive the set of equilibria without artificially altering the sharing rule or relying on elegantly complex mixed strategies (Blume, 2003). In the Hotelling location model, if there are a finite number of locations (each of which is populated) we can show that the equilibria are the unique strategy(ies) to survive iterated deletion of dominated strategies.

2 The Bertrand Game

This model was developed as a critique of the Cournot model, in his famous 1888 review Bertrand said:

> Indeed, whatever the common price adopted, if one of the owners, alone, reduces his price, he will, ignoring any minor exceptions, attract all the buyers, and thus double his revenue if his rival lets him do so. (de Bornier, 1992)[1]

Later on this insight was formally developed into what is now called the Bertrand game.

2.1 Model

There are two firms with constant marginal costs choosing price to maximize their profits. Denote the marginal costs of firm i as $c_i \geq 0$. We restrict this analysis by assuming that price must be in units—κ—and for simplicity that all other variables are as well. The market demand curve is $D(p)$ which is downward sloping ($D' < 0$), based on this and the prices of the two firms (p_1, p_2) firm one's demand curve is:

$$d_1(p_1, p_2) = \begin{cases} 0 & \text{if} \quad p_1 > p_2 \\ \frac{1}{2}D(p_1) & \text{if} \quad p_1 = p_2 \\ D(p_1) & \text{if} \quad p_1 < p_2 \end{cases}.$$

thus firm one maximizes $\pi_1(p_1, p_2) = (p_1 - c_1)\, d_1(p_1, p_2)$.

[1] We must take issue with this argument.

Imagine that two hot dog sellers were side by side. One of them was charging 4.99 and the other 5.00. Would you categorically and always go to the cheaper firm? I doubt most would, and he explicitly states *almost all* the customers will switch. This might happen if the cheaper firm offered a 33% discount.

2.2 The Best Response

In this section we will always analyze things from firm 1's point of view without loss of generality. The *monopoly price* will be denoted p_1^m:

$$p_1^m \equiv \arg\max_p (p - c_1) D(p)$$

and please note that this will be in κ units. If the other firm (for any reason) prices strictly higher than this they have chosen not to compete and firm one should simply ignore them, thus if $p_2 > p_1^m$ then the best response is p_1^m.

With that out of the way we can immediately derive Bertrand's core insight, undercutting.

Lemma 1 (Undercutting) *If $p_1^m \geq p_2 \geq c_1 + 2\kappa$ then $p_1 = p_2 - \kappa$ is the best response.*

Proof. *We first notice that $\pi_1(p_1, p_2)$ is strictly positive and increasing in p_1 over this range. Thus we know charging a strictly higher price cannot be optimal. Likewise $p_2 - m\kappa$ will give a lower profit than $p_2 - \kappa$ for any $m > 1$. Thus the only candidates for best responses are $p_1 = p_2$ and $p_1 = p_2 - \kappa$. The corresponding profits are:*

$$\begin{aligned} \pi_1(p_2, p_2) &= (p_2 - c_1) \frac{1}{2} D(p_2) \\ \pi_1(p_2 - \kappa, p_2) &= (p_2 - \kappa - c_1) D(p_2 - \kappa) \end{aligned}$$

we hypothesize that $\pi_1(p_2 - \kappa, p_2) > \pi_1(p_2, p_2)$, and notice that $D(p_2 - \kappa) > D(p_2)$ because demand is strictly downward sloping. Since $p_2 - c_1 > p_2 - \kappa - c_1 > \kappa > 0$ our hypothesis will be verified if:

$$(p_2 - \kappa - c_1) \geq (p_2 - c_1) \frac{1}{2}$$

$$\begin{aligned} 2(p_2 - \kappa - c_1) &\geq (p_2 - c_1) \\ p_2 &\geq -c_1 + 2\kappa + 2c_1 \\ p_2 &\geq c_1 + 2\kappa. \end{aligned}$$

or the range above. ■

At this point it is worthwhile dropping into a case by case basis. If $p_2 = c_1 + \kappa$ then firm one will get a strictly positive profit by charging $p_1 = p_2$, while anything strictly lower or higher will result in at best a zero profit. If $p_2 = c_1$ then firm one clearly does not want to charge a lower price, on the other hand if they match the price they get zero profit. But, one might notice, this would also be true if they charged a strictly higher price, so $BR_1(c_1) = c_1 + m\kappa$ for $m \in \mathbb{N} \equiv (0, 1, 2, ...)$.

If the other firm is pricing below our marginal cost, $p_2 < c_1$ then firm 1 wants to price higher than them. That will guarantee zero profits while matching or charging a lower price will give negative profits. Thus if $p_2 < c_1$ $BR_1(p_2) = p_2 + (m + 1)\kappa$. Thus we conclude:

Proposition 2 *If all variables are in κ units then the best response of firm one is:*

$$BR_1(p_2) = \begin{cases} p_1^m & \text{if} \quad p_2 > p_1^m \\ p_2 - \kappa & \text{if} \quad p_1^m \geq p_2 \geq c_1 + 2\kappa \\ p_2 & \text{if} \quad p_2 = c_1 + \kappa \\ p_2 + m\kappa & \text{if} \quad p_2 = c_1 \\ p_2 + (m+1)\kappa & \text{if} \quad p_2 < c_1 \end{cases}$$

for $m \in \mathbb{N}$.

Some readers might take issue with pricing below your marginal cost, after all what if the other firm makes a mistake? For these readers I point out that a best response is about the *possible* optimal strategies. You might, for various reasons, prefer one or the other of these in practice.

2.2.1 Equilibria

There are two cases to consider, first of all the simple and direct one where $c_1 = c_2 = c$ or the two firms are symmetric. In this case the pure strategy equilibria are $p_1 = p_2 \in \{c, c+\kappa\}$. If $p_1 = c + \kappa$ then as we have explained the best response of firm 2 is $p_2 = p_1$, thus this is a Nash equilibrium. If $p_1 = c$ then above we argued that firm 2 *could* charge $p_2 = p_1$, but they could also charge any strictly higher price. If $p_2^* \geq c + 2\kappa$ it will unleash a cycle of price undercutting, namely $p_1 = p_2^* - \kappa$ and then $p_2 = p_2^* - 2\kappa$, and so on until $\max(p_1, p_2) < c + 2\kappa$. In this equilibrium $p_2 = c$ to stop such cycles from occurring.[2]

While it is outside the scope of this paper, one might wonder about mixed strategy equilibria. Interestingly enough their do not seem to be any, as is proven in the appendix. This is surprising because in general there is an odd number of Nash equilibria, but this is a generic property (Govindan and Wilson, 2001).

The second case is worthy of a lemma, even if the result is fairly straightforward.

Lemma 3 *If $c_1 < c_2$ then in equilibrium $p_2 = p_1 + \kappa$ for $c_1 \leq p_1 \leq c_2$.*
Proof. *If $c_1 \leq p_1 < c_2$ then firm 2 wants to charge a strictly higher price, and the least higher price they can charge is $p_2 = p_1 + \kappa$. If they choose $p_2 > p_1 + \kappa$ then firm 1 will want to increase their price, which cannot be true in an equilibrium. If $p_1 = c_2$ then firm 2 would be willing to match the price, but then firm 1 would want to strictly decrease their price. Thus if $p_1 = c_2$ p_2 must be $c_2 + \kappa$ to be sure that firm 1 does not want to raise or lower their price. If $p_1 = c_2 + \kappa$ then firm 2 will match them, then firm one will want to lower their price. For any $p_1 \geq c_2 + 2\kappa$ firm 2 will want to undercut their price. Thus this is the set of equilibria.* ■

In the continuous model these equilibria are usually characterized by $p_2 = p_1$ *but firm 2 gets no demand.* This is inconsistent, in our model we asserted that

[2] The reader might be amused to realize that $p_2 = c$ is actually a weakly dominated strategy in this game.

if they charged the same price they would split demand but now we declare that the consumers only go to one of the firms. One can solve this conundrum with mixed strategies, as was argued in Bloom (2003). However that seems an extreme solution to a problem that does not occur in our simpler model.

3 Hotelling Location Model

The classical Hotelling location model has two ice cream sellers choosing a location on a beach with a uniform distribution of customers. This is restrictive because it does not discuss what happens if the distribution is not uniform, and in general the best responses do not exist because the two sellers will want to be close to each other. Our finite model does not depend on the distribution (other than it being strictly positive) and, of course, always has a best response.

3.1 The Model

Two firms compete by choosing location $l_i \in (1, 2, 3, ...L)$. They charge the same price that is strictly above marginal cost, and thus their objective is to maximize their number of customers.

At each location $l \in (1, 2, 3, ...L)$ there will be $c_l > 0$ customers. Usually they will always go to the closest firm, if the two firms are an equal distance from their location they will split their demand. For distance we use the standard Euclidian metric, $d(l, l') = (l - l')^2$, notice that this means the space is linear. If one wishes one can assume c_l is always even, but it equally well could be a mass. The total number of customers is $C = \sum_{l=1}^{L} c_l$.

The assumption that every location has some customers is critical to our analysis. It would not change the equilibria, but it would certainly affect our argument for iterated deletion of dominated strategies. Of course one could equally well delete locations with zero customers, but likewise one could add as many of them as one likes.

3.2 The Best Response

Our first result is the critical intuition, you want to be as close to your rival as possible. In this game if your opponent is at a higher location than you then all the customers at a lower location are locked into going to you. Locking in as many as possible means you want to be as close to your rival as possible. Like before we will analyze the game from firm one's perspective without loss of generality.

Lemma 4 (Business Stealing) $BR_1(l_2) \in \{l_2 - 1, l_2, l_2 + 1\}$.
Proof. *The only way this could be false is if* $BR_1(l_2) = l_2 \pm m$ *for* $m > 1$*, without loss of generality consider* $l_2 + m$*. We wish to show that* $l_2 + m - 1$ *will result in a strictly larger number of customers. To do this consider the location* $l_2 + \left\lfloor \frac{m}{2} \right\rfloor$*. If* m *is odd these customers all go to firm* 2*, if* m *is even half of them*

go to firm 2. If firm one chooses the location $l_2 + m - 1$ either half or all of these customers will go to firm 1, strictly increasing their sales. ■

We can be even more precise if one defines the *median location(s).* This set, $\mathcal{L}_m$, are the location or locations for which half the customers are located above or below that location: $l^* \in \mathcal{L}_m$ if $\sum_{l=1}^{l^*} c_l \geq \frac{C}{2}$ and $\sum_{l=l^*}^{L} c_l \geq \frac{C}{2}$. We notice that either there is a unique median location or there are two consecutive locations, and denote $\lceil \mathcal{L}_m \rceil$ its highest member and $\lfloor \mathcal{L}_m \rfloor$ its lowest. With this definition we find:

Proposition 5 *The best response is:*

$$BR_1(l_2) = \begin{cases} l_2 - 1 & \text{if} \quad l_2 > \lceil \mathcal{L}_m \rceil \\ l^* & \text{if} \quad l_2 \in \mathcal{L}_m \text{ for } l^* \in \mathcal{L}_m \\ l_2 + 1 & \text{if} \quad l_2 < \lfloor \mathcal{L}_m \rfloor \end{cases}$$

Proof. *Consider the case $l_2 > \lceil \mathcal{L}_m \rceil$ if $l_1 > l_2$ then firm 2 will get at least $C/2$ customers, and thus firm one will get less than half. If $l_1 < l_2$ firm 1 will get at least $C/2$ customers, thus this is optimal. A symmetric argument holds if $l_2 < \lfloor \mathcal{L}_m \rfloor$. If $l_2 \in \mathcal{L}_m$ then any $l^* \in \mathcal{L}_m$ will give exactly $C/2$ customers, and $l_1 > \lceil \mathcal{L}_m \rceil$ or $l_1 < \lfloor \mathcal{L}_m \rfloor$ will give strictly less than that, thus it is optimal.* ■

3.3 The Equilibrium

In this game if there are two median locations then there can be many equilibria. However the critical point is that all of them will have the same payoff; there is only one outcome.

This outcome is that each of them get half the customers and both locate at the (or a) median location. The last proposition argued that only at these locations would no one want to change their location, and that is required in a Nash equilibrium.

3.4 The Outcome of Iterated Deletion of Dominated Strategies

This result actually only relies on the common knowledge of rationality. Showing this means we need to show that some strategies are dominated, this requires a case by case analysis for each pair of strategies, and in general can be onerous. In this game it is fairly straightforward.

For this lemma let $\bar{L}$ be the highest strategy that has not already been deleted by iterated deletion of dominated strategies.

Lemma 6 *If $\bar{L} > \lceil \mathcal{L}_m \rceil$ then $l_1 = \bar{L} - 1$ dominates $l_1 = \bar{L}$.*
Proof. *First assume that $l_2 < \bar{L} - 1$ then like in Lemma 4 $l_1 = \bar{L} - 1$ will get strictly more customers than $l_1 = \bar{L}$. If $l_2 = \bar{L} - 1$ then $l_1 = \bar{L} - 1$ will give $C/2$ customers and $l_1 = \bar{L}$ will give strictly less. If $l_2 = \bar{L}$ then if $l_1 = \bar{L} - 1$*

they will get strictly more than $C/2$ customers since they have captured all the customers in the median location(s), while $l_1 = \bar{L}$ will given $C/2$. Thus for any l_2 $l_1 = \bar{L} - 1$ gives a strictly higher payoff than $l_1 = \bar{L}$ and dominates it. ■

If $\underline{L}$ is the least location that has not been deleted, a symmetric argument holds for it. Since these locations are dominated, we can conclude that rational firms will never choose them, and drop them from further analysis. We repeat this until $\bar{L} \leq \lceil \mathcal{L}_m \rceil$ and $\underline{L} \geq \lfloor \mathcal{L}_m \rfloor$.

4 Appendix: Mixed strategy equilibria in the Bertrand game.

Lemma 7 *If $c_1 = c_2 = c$ then there are no non-degenerate mixed strategy equilibria in the Bertrand game*
Proof. *Let $\bar{p}$ be the highest price in the support of the mixed strategy equilibrium, and assume it is played with probability $\rho_{\bar{p}} > 0$. We first consider the case where $\bar{p} \geq c + 2\kappa$. Consider $\bar{p} - \kappa$ which is played with probability $\rho_{\bar{p}-\kappa}$ which might be zero. Then the expected payoff of these two strategies are:*

$$
\begin{aligned}
E\pi(\bar{p}) &= \rho_{\bar{p}}(\bar{p} - c)\frac{1}{2}D(\bar{p}) \\
E\pi(\bar{p} - \kappa) &= \rho_{\bar{p}}(\bar{p} - \kappa - c)\frac{1}{2}D(\bar{p} - \kappa) + \rho_{\bar{p}-\kappa}(\bar{p} - \kappa - c)\frac{1}{2}D(\bar{p} - \kappa)
\end{aligned}
$$

and Lemma 1 shows that $(\bar{p} - c)\frac{1}{2}D(\bar{p}) < (\bar{p} - \kappa - c)\frac{1}{2}D(\bar{p} - \kappa)$ thus $\bar{p}$ is not a best response. Now consider $\bar{p} = c + \kappa$. Notice that playing $c + \kappa$ will result in sometimes getting a strictly positive profit, while playing any lower price will at best result in zero profits, thus if $\rho_{c+\kappa} > 0$ the unique best response is $c + \kappa$. ■

5 References

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