The Hotelling Location Model with a Discrete Number of Locations.

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1 Introduction and Model

There are many reasons that firm location is important. There is, of course, physical location but to be frank location in the taste spectrum is more important. Take, for example, Coca Cola and Pepsi. While I am sure you have a precise brand you prefer (Cola Zero for me thank you) you must admit they taste very similar. Why? Why do they not differentiate themselves more? In the land of caffeinated beverages/energy drinks there is a large variety—but again they seem to be clustered around a few locations. Some firms (Starbucks for example) work hard to produce a large variety of energy drinks, work hard to make sure they can satisfy a variety of tastes. Why is this not more common?

Hotelling was the first to address this question. He considered a continuum of customers uniformly distributed over a linear space—the unit interval. Like is often the case this means that best responses do not exist outside of equilibrium. Because of this I began investigating the model with a discrete number of locations, and not only are the results more general (they hold for any distribution) but they are both simple and elegant. Thus I will show you these results and how to derive them in this handout.

1.1 Model

There are two firms, a and b, who choose their location $l_x \in \{1, 2, 3, .., L\}$ for $x \in \{a, b\}$. For convenience we will always assume L is odd. These firms sell identical products at identical prices, which is fixed above marginal cost.

A customer will have a location $y \in \{1, 2, 3, .., L\}$ and will buy from firm a if $d(l_a, y) < d(l_b, y)$ and if $d(l_a, y) = d(l_b, y)$ they will buy from firm a half the time. The distance metric can be the quadratic $d(l_a, y) = (l_a - y)^2$, it will not matter that much. Thus the appropriate description of the distribution of customers the total number of customers at each location, this is c_l for $l \in \{1, 2, 3, .., L\}$. We assume $c_l > 0$ and let $C = \sum_{l=1}^{L} c_l$ be the total number of customers.

Let $D_a(l_a, l_b) = \sum_{l=1}^{L} c_l 1_{d(l_a,l)<d(l_b,l)} + \frac{1}{2} \sum_{l=1}^{L} c_l 1_{d(l_a,l)=d(l_b,l)}$ where 1_x is one if x is true and zero otherwise. Then their profit function is:

$$\max_{l_a} P D_a(l_a, l_b) - C(D_a(l_a, l_b)) \tag{1}$$

and the derivative with respect to $D_a(l_a, l_b)$ is $P - MC(D_a(l_a, l_b)) > 0$ by assumption. Thus the objective can be simply written as:

$$\max_{l_a} D_a\left(l_a, l_b\right) \tag{2}$$

and this is what we will analyze.

2 An Example

Let $L = 7$ and the distribution of customers be:

Location	1	2	3	4	5	6	7	
c_l	10	2	4	6	4	12	8	$C = 46$

(3)

then the easiest way to solve this is to fill out a table. THe columns will be the locations of firm b, and the rows will be the locations of firm a, and then in each square we will write $D_a\left(l_a, l_b\right)$. Notice that $D_a\left(l_a, l_b\right) + D_b\left(l_a, l_b\right) = C$ and that $D_a\left(x, y\right) = D_b\left(y, x\right)$, this will greatly simplify this analysis. For example if $l_a = l_b$ $D_a = \frac{C}{2}$. The table is as follows:

$l_b =$	1	2	3	4	5	6	7
$l_a = 1$	23	10	11	12	14	16	19
$l_a = 2$	$\underline{36}$	23	12	14	16	19	22
$l_a = 3$	35	$\underline{34}$	23	16	19	22	24
$l_a = 4$	34	32	$\underline{30}$	23	22	24	26
$l_a = 5$	32	30	27	$\underline{24}$	$\underline{23}$	$\underline{26}$	32
$l_a = 6$	30	27	24	22	20	23	$\underline{34}$
$l_a = 7$	27	24	22	20	14	8	23

(4)

Firm a's best responses then are the maximum of demand in a given column:

$l_b =$	1	2	3	4	5	6	7
$BR_a\left(l_b\right)$	2	3	4	5	5	5	6

(5)

and once we recognize that $BR_a\left(x\right) = BR_b\left(x\right)$ (symmetry) we can immediately realize that the unique Nash equilibrium is $l_a = l_b = 5$.

There are quite a few interesting facts here. First of all, define $l_m \in \{1, 2, 3, .., L\}$ as a location such that $\sum_{l=1}^{l_m} c_l \geq \frac{C}{2}$ and $\sum_{l=l_m}^{L} c_l \geq \frac{C}{2}$. This is the *median location*, in this example this is 5, and the Nash equilibrium is to have both firms locate at the median location. Second is "business stealing," the best response to l_b is always either l_b or "right next to it" at either $l_b - 1$ or $l_b + 1$. In other words they want to be very close to their competitor to steal as many customers as possible. Third—expanding on the second point—the best response to l_b is $l_b + 1$ if $l_b < l_m$, and $l_b - 1$ if $l_b > l_m$.

Finally note we actually can solve this model using iterated deletion of dominated strategies. Notice that $D_a\left(6, l_b\right) > D_a\left(7, l_b\right)$ and $D_a\left(2, l_b\right) > D_a\left(1, l_b\right)$.

This means locations 6 dominates location 7 and 2 dominates location 1. Since no sensible firm would locate in a dominated location (6 is always a strictly better plan than 7) we can eliminate locations 1 and 7 from analysis. But then 3 dominates 2 and 5 dominates 6. We stop going down from above now, but we can still show that—after eliminating 2—4 dominates 3, and then after eliminating 3 we can see 5 dominates 4. This is a very powerful result, not only is 5 the Nash equilibrium (which requires both rationality and correct expectations) but it is the unique strategy that survives iterated deletion of dominated strategies (which requires only common knowledge of rationality). Is this general? We will show that with some mild assumptions it is.

3 General Analysis

First let us lay out our assumptions. Generally speaking these assumptions are just for simplicity, if we do not make them analysis is just more complicated—however the proof of iterated deletion of dominated strategies does strictly rely on $c_l > 0$ for all l.

Assumptions $c_l > 0$ for all $l \in \{1, 2, 3, .., L\}$, and l_m is unique.

3.1 Best Responses and Nash Equilibrium

We can immediately generalize the insights in our example.

Lemma 1 (Business Stealing) $BR_a(l_b) \in \{l_b - 1, l_b, l_b + 1\}$
Proof. *We only rule out* $BR_a(l_b) = l_b \pm k$ *for* $k > 1$*, thus let us consider locating at* $l_b + k$*. This means that there are* $\sum_{l=l_b+1}^{l_b+(k-1)} c_l > 0$ *customers in between firms a and b, at least some of these will go to firm b, if firm a chooses* $l_b + 1$ *instead of* $l_b + k$ *all those customers will go to firm a, giving them a strictly higher payoff.* ■

Lemma 2 (Move to the Median)

$$BR_a(l_b) = \begin{cases} l_b + 1 & \text{if} \quad l_b < l_m \\ l_b & \text{if} \quad l_b = l_m \\ l_b - 1 & \text{if} \quad l_b > l_m \end{cases}$$

Proof. *To recognize this we must first realize that at a best response* $D_a(l_a, l_b) \geq \frac{C}{2}$*. This is because if* $l_a = l_b$ *then* $D_a(l_a, l_b) = \frac{C}{2}$ *so at any location firm one can achieve* $\frac{C}{2}$*. If* $l_a = l_b - 1$ *then* $D_a(l_b - 1, l_b) = \sum_{l=1}^{l_b-1} c_l$ *and this is only greater than* $\frac{C}{2}$ *if* $l_b > l_m$*. A symmetric argument shows that if* $l_b < l_m$ *then* $l_a = l_b + 1$*. If* $l_b = l_m$ *then since we assumed* l_m *was unique* $\sum_{l=1}^{l_b-1} c_l < \frac{C}{2}$ *and* $\sum_{l=l_b+1}^{L} c_l < \frac{C}{2}$ *thus the best response is established.* ■

Noticing the symmetry of the best responses we can immediately conclude:

Proposition 3 *If $c_l > 0$ for all l and l_m is unique then the unique Nash equilibrium of the Hotelling location game is $l_a = l_b = l_m$.*

3.2 Dominance and Iterated Dominance

We say that one strategy dominates the other if it gives a strictly higher payoff (is a *better* response) no matter what the other player(s) does. Obviously, logically, no rational player would use a dominated strategy. Thus if we assume common knowledge of rationality then each player can assume that the others would not use a dominated strategy, and we can iterate this concept. In this game we can iteratively delete the highest and the lowest strategy, until the only strategy we are left with is the median location.

A warning, this proof strictly requires $c_l > 0$ for all l. If this condition is violated then we would only have "weakly dominant" in the following proof. Weak dominance is not a valid reason to remove a strategy, for example in the classic Bertrand Duopoly one can easily show that the only Nash equilibrium is in weakly dominated strategies. (If $p = c$ then your profits will always be zero, while if $p > c$ then your profits will sometimes be strictly positive. Thus the only Nash equilibrium ($p_1 = p_2 = c$) is weakly dominated by any $p > c$.)

Lemma 4 *Let $\underline{l}$ be the minimal location under consideration, and $\bar{l}$ be the maximal location. Then if $\underline{l} < l_m$ $\underline{l}+1$ dominates $\underline{l}$, and if $\bar{l} > l_m$ then $\bar{l}-1$ dominates $\bar{l}$.*
Proof. *We must show that $D_a(\underline{l}+1, l_b) > D_a(\underline{l}, l_b)$ for every $l_b \in \{\underline{l}, \underline{l}+1, \underline{l}+2, ..., \bar{l}\}$. First if $l_b = \underline{l}$ then $D_a(\underline{l}, \underline{l}) = \frac{C}{2}$ and $D_a(\underline{l}+1, \underline{l}) = \sum_{l=\underline{l}+1}^{L} c_l > \frac{C}{2}$ since $\underline{l} < l_m$. If $l_b = \underline{l}+1$ then this also shows that $D_a(\underline{l}, \underline{l}+1) < \frac{C}{2} = D_a(\underline{l}+1, \underline{l}+1)$. Now consider $l_b = \underline{l}+k$ for $k > 1$ then $D_a(\underline{l}, l_b) + \frac{1}{2}c_{\underline{l}+1} \leq D_a(\underline{l}+1, l_b)$ since at the very least firm a will capture half of the customers at $c_{\underline{l}+1}$ if $l_a = \underline{l}+1$. Thus the proof is done, a symmetric argument easily establishes that $\bar{l}-1$ dominates $\bar{l}$.* ■

We can now iterate this concept and conclude:

Proposition 5 *If $c_l > 0$ for all l and l_m is unique, then it is also the unique strategy to survive iterated deletion of dominated strategies.*
Proof. *The proof above shows that if $\underline{l} < l_m$ $\underline{l}+1$ dominates $\underline{l}$ when $l_b \in \{\underline{l}, \underline{l}+1, \underline{l}+2, ..., \bar{l}\}$ and likewise $\bar{l} > l_m$ then $\bar{l}-1$ dominates $\bar{l}$ when $l_b \in \{\underline{l}, \underline{l}+1, \underline{l}+2, ..., \bar{l}\}$, thus we can immediately iterate it and the process will stop from either direction when $\underline{l} = l_m$ or $\bar{l} = l_m$.* ■

Notice that generally speaking this process will stop after a different number of iterations from each direction. In the example above $l_m = 5$ so it only took two steps to get from 7 to 5, and four steps coming from below.

4 Welfare and Generalization

In welfare terms this equilibrium is just awful. As long as the welfare function wants to minimize the distance between customers and the firms it would clearly be welfare dominant to have the two firms at different location. More intuitively this result is just ridiculous, consider the following distribution for any natural number $k \in \{0, 1, 2, ...\}$.

$$\begin{array}{llll} \text{Location} & 1 & 2 & 3 \\ c_l & 10^k & 2 & 10^k \end{array} \tag{6}$$

If $k = 0$ then maybe you might want both firms at location 2, but what if $k = 6$, $10^6 = 1,000,000$? Surely a sensible planner would insist that one firm locate at location 1 and one locate at location 3? What if $k = 15$?

In terms of general insight, well this seems very sound. We often observe chain stores congregated in one area in new shopping malls, etcetera. However while this result suggests why they might do that in the model the result only holds if there are two firms. If there are three or more firms there might be mixed strategy equilibria and possibly many different equilibria. For example in our example (displayed item 3) we can find an 4 step cycle in pure strategy best responses, thus proving a mixed strategy equilibrium exists.

Location	1	2	3	4	5	6	7
c_l	10	2	4	6	4	12	8
Initial distribution				l_b	l_c	l_a	
$BR_c(l_a, l_b)$			l_c	l_b		l_a	
$BR_a(l_b, l_c)$			l_c	l_b	l_a		
$BR_b(l_a, l_c)$			l_c		l_a	l_b	
$BR_c(l_a, l_b)$				l_c	l_a	l_b	

(7)