ECON 439
Final: Extensive Form Games

Kevin Hasker

This exam will start at 18:40 and finish at 20:20.

Answer all questions in the space provided. Points will only be given for work shown.

1. *(12 points)* Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test. I recognize this means I should not use calculators or other electronic devices.

   Name and Surname: _

   Student ID: _

   Signature:

   _

μ	η	γ	κ	χ	$w(e) \in$	$e_h^* \in$	$e^* \in$
$\frac{2}{3}$	20	2	3	6	$\{2, 14, 20\}$	$[3, 6]$	$[0, 2]$
$\frac{1}{2}$	20	4	2	4	$\{4, 12, 20\}$	$[4, 8]$	$[0, 2]$
$\frac{3}{4}$	18	6	2	3	$\{6, 15, 18\}$	$[4, 6]$	$[0, 3]$
$\frac{1}{l}p$	$\gamma + \Delta$	γ	$\frac{1}{h}\Delta$	$\frac{1}{l}\Delta$	$\left\{\gamma, \gamma + \frac{1}{l}p\Delta, \Delta + \gamma\right\}$	$[l, h]$	$[0, p]$

2. *(35 points total)* Consider the Spence signalling model of education. There is one worker and multiple firms. The worker may be high productivity with probability $\lambda = \mu$—in which case they are worth $\pi_h = \eta$ to a firm—or they may be low quality—in which case they are worth $\pi_l = \gamma$ to the firm. The worker may choose to get education, $e \in [0, \infty)$. This education has no impact on their value to the firm, and has a marginal cost of $c_h = \kappa$ if the worker is high quality and $c_l = \chi$ if the worker is low quality. The timing of the game is as follows:

   0. Nature chooses the worker's type.
   1. The worker chooses her education level.
   2. The firms compete for the worker.
   3. The worker chooses which—if any—firm she will work for.

   The worker's utility of choosing (w, e) is $u_x(w, e) = w - c_x e$ where $x \in \{l, h\}$ and the firm's profit is of hiring a worker of type x is $\pi = \pi_x - w$.

   We are interested in *self-selection equilibria* which is a (w_h, e_h, w_l, e_l) where type h chooses (w_h, e_h) and type l chooses (w_l, e_l)—these must also be weak sequential equilibria.

   (a) *(3 points)* Assume that some type of worker chooses an e, what are the possible values for $w(e)$ and why?

Solution 1 $w(e) \in \{\gamma, \mu\eta + (1-\mu)\gamma, \eta\}$. *This is because the firms must compete by offering a wage, this Bertrand style competition will result in zero expected profits. So they can either know the worker's type based on e or assume they are part of the general pool. (Please note I am only considering pure strategy equilibria, which is all we are analyzing unless otherwise noted.)*

(b) *(8 points)* Write down the four constraints on the worker in a self-selection equilibrium.

Solution 2 *These are the Incentive compatibility constraints (IC) and the individual rationality constraints (IR). Mathematically these are:*

$$\begin{aligned} IC_H &: w_h - \kappa e_h \geq w_l - \kappa e_l \\ IC_L &: w_l - \chi e_l \geq w_h - \chi e_h \\ IR_H &: w_h - \kappa e_h \geq \gamma \\ IR_L &: w_l - \chi e_l \geq \gamma \end{aligned}$$

(c) *(4 points)* Which of these four constraints is unnecessary? Prove that it is implied by the other constraints.

Solution 3 *The constraint IR_H is unnecessary, it is implied by IC_H, $c_h < c_l$, and IR_L.*

$$\begin{aligned} IC_H &: w_h - \kappa e_h \geq w_l - \kappa e_l \\ w_l - \kappa e_l &\geq w_l - \chi e_l \\ IR_L &: w_l - \chi e_l \geq \gamma \end{aligned}$$

thus the combination is $w_h - \kappa e_h \geq \gamma$, or IR_H.

(d) *(9 points total)* If $e_h \neq e_l$:

i. *(2 points)* What will be the value of e_l?

Solution 4 *Since choosing e_l reveals the worker is low productivity $w_l = \gamma$ and then IR_L implies:*

$$\begin{aligned} IR_L &: \gamma - \chi e_l \geq \gamma \Rightarrow \\ e_l &\leq 0 \end{aligned}$$

thus $e_l = 0$.

ii. *(4 points)* What will be the range of values possible for e_h?

Solution 5 $w_h = \eta$ *thus our remaining constraints are:*

$$\begin{aligned} IC_H &: \eta - \kappa e_h \geq \gamma \\ IC_L &: \gamma \geq \eta - \chi e_h \end{aligned}$$

the first one implies $e_h \leq \frac{\eta-\gamma}{\kappa}$ the second implies that $e_h \geq \frac{\eta-\gamma}{\chi}$, thus the range is $\left[\frac{\eta-\gamma}{\chi}, \frac{\eta-\gamma}{\kappa}\right]$.

iii. *(3 points)* What type of equilibria are these? Who is signalling in these equilibria?

Solution 6 *These are* separating *equilibria, and the high quality worker is signalling that they are high quality.*

(e) *(7 points total)* If $e_h = e_l = e^*$:

i. *(4 points)* What will be the range of values possible for e^*?

Solution 7 *Obviously we have loss all of the incentive compatibility constraints, leaving us only IR_L. Since everyone is choosing the same level of education $w = \mu\eta + (1-\mu)\gamma$, thus our constraint is:*

$$\begin{aligned} \mu\eta + (1-\mu)\gamma - \chi e^* &\geq \gamma \\ \mu(\eta-\gamma) &\geq \chi e^* \\ \frac{\mu(\eta-\gamma)}{\chi} &\geq e^* \end{aligned}$$

thus the range is $\left[0, \frac{\mu(\eta-\gamma)}{\chi}\right]$

ii. *(3 points)* What type of equilibria are these? Who is signalling in these equilibria?

Solution 8 *These are* pooling *equilibria, in these equilibria (if $e^* > 0$) the low types are trying to signal that they are average, or deceive the firm.*

(f) *(4 points)* Give a general definition of a signal and find the unique equilibrium in this game where there is no signalling.

Solution 9 *A* signal *is any action for which the direct marginal benefit is less than the direct marginal cost. This action is taken to hide or reveal information about some underlying type of the player.*
The unique equilibrium without signalling is the pooling equilibrium where no one goes to school. ($w = \mu\eta + (1-\mu)\gamma$, $e^* = 0$).

3. *(28 points total)* Consider the following game as the stage game of a T period repeated game (where $T < \infty$ or $T = \infty$.)

Player 1 \ Player 2		α	β	ε
Player 1	A	$a+2b; b-c$	$-b; b-c$	$a; b$
	B	$a+b; b+c$	$-b; c-a$	$a-c; b$
	C	$a-b; -b$	$0; 0$	$a+2b; -c$

Player 1 \ Player 2		α	β	ε
Player 1	A	$10; 2^1$	$-3; 2$	$4; 3^2$
	B	$7; 4^2$	$-3; -3$	$3; 3$
	C	$1; -3$	$0; 0^{12}$	$10; -1^1$

(X,χ)	δ^*	$\bar{\delta}$	a	b	c
(A,ε)	$\frac{3}{5}$	$\frac{3}{7}$	4	3	1

Player 2

		α	β	ε
Player 1	A	$a-b;-b$	$a+2b;-c$	$0;0$
	B	$a+b;b+c$	$a-c;b$	$-b;c-a$
	C	$a+2b;b-c$	$a;b$	$-b;b-c$

Player 2

		α	β	ε
Player 1	A	$2;-1$	$5;-7^1$	$0;0^{12}$
	B	$4;8^2$	$-4;1$	$-1;4$
	C	$5;-6^1$	$3;1^2$	$-1;-6$

(X,χ)	δ^*	$\bar{\delta}$	a	b	c
(C,β)	$\frac{2}{5}$	$\frac{1}{4}$	3	1	7

Player 2

		α	β	ε
Player 1	A	$a+2b;b-c$	$-b;b-c$	$a;b$
	B	$a-b;-b$	$0;0$	$a+2b;-c$
	C	$a+b;b+c$	$-b;c-a$	$a-c;b$

Player 2

		α	β	ε
Player 1	A	$10;-1^1$	$-4;-1$	$2;4^2$
	B	$-2;-4$	$0;0^{12}$	$10;-5^1$
	C	$6;9^2$	$-4;3$	$-3;4$

(X,χ)	δ^*	$\bar{\delta}$	a	b	c
(A,ε)	$\frac{4}{5}$	$\frac{2}{3}$	2	4	5

(a) *(8 points)* Find the best responses and all of the Nash equilibria of this stage game. You may mark your best responses above but explain your notation below.

Solution 10 *Like usual I have marked a 1 in the upper right hand corner if it is a best response for player 1, and a 2 for player 2. The unique NE is the one box with both a 1 and a 2 in the upper right hand corner. In the game:*

Player 2

		α	β	ε
Player 1	A	$10;2^1$	$-3;2$	$4;3^2$
	B	$7;4^2$	$-3;-3$	$3;3$
	C	$1;-3$	$0;0^{12}$	$10;-1^1$

it is (C,β).

(b) *(3 points)* Find the *pure strategy* minimax strategies in this game.

Solution 11 *Using the game:*

		Player 2 α	β	ε
	A	$a+2b; b-c$	$-b; b-c$	$a; b$
Player 1	B	$a+b; b+c$	$-b; c-a$	$a-c; b$
	C	$a-b; -b$	$0; 0$	$a+2b; -c$

$u_1(BR_1(\alpha), \alpha) = a + 2b$, $u_1(BR_1(\beta), \beta) = 0$, $u_1(BR_1(\varepsilon), \varepsilon) = a + 2b$, *thus the minimax is* (C, β)

$u_2(A, BR_2(A)) = b$, $u_2(B, BR_2(B)) = b + c$, $u_2(C, BR_2(C)) = 0$, *thus the minimax is* (C, β)

(c) *(6 points)* Assume $T < \infty$, or the finite repeated game. Characterize the set of equilibrium payoffs for all T. Prove your result. *Be careful to prove your result in every subgame.*

Solution 12 *The set of equilibrium payoffs is* $(0, 0)$.
To explain why, I will use the game:

		Player 2 α	β	ε
	A	$a+2b; b-c$	$-b; b-c$	$a; b$
Player 1	B	$a+b; b+c$	$-b; c-a$	$a-c; b$
	C	$a-b; -b$	$0; 0$	$a+2b; -c$

In the final period, the past does not matter (it only adds a constant to the payoff) so the only viable prediction is the unique NE of the stage game, or (C, β).

In the next to last period, the future now will always be (C, β) *regardless of what occurs today (this only adds a constant to payoffs) thus the only viable prediction is* (C, β).

The last step can be iterated to show that the subgame perfect equilibrium in the game is (C, β) *in every period, which will result in a per-period and total payoff of zero.*

(d) *(2 points)* For $x \in \{A, B, C\}$ and $y \in \{\alpha, \beta, \varepsilon\}$ show that the value of playing (x, y) in every period is $u_i(x, y) / (1 - \delta)$, where $\delta \in (0, 1)$ is the discount factor.

Solution 13 *Let this stream just be denoted* (x,y) *then*

$$\begin{aligned} V_i(x,y) &= u_i(x,y) + \delta u_i(x,y) + \delta^2 u_i(x,y) + \delta^3 u_i(x,y) + ... \\ \frac{V_i(x,y)}{u_i(x,y)} &= 1 + \delta + \delta^2 + \delta^3 + ... \\ (1-\delta)\frac{V_i(x,y)}{u_i(x,y)} &= (1-\delta) + \left(\delta - \delta^2\right) + \left(\delta^2 - \delta^3\right) + \left(\delta^3 - \delta^4\right) + ... \\ &= 1(-\delta+\delta)\left(-\delta^2+\delta^2\right)\left(-\delta^3+\delta^3\right) - \delta^4 + ... \\ &= 1 \\ V_i(x,y) &= \frac{u_i(x,y)}{1-\delta} \end{aligned}$$

(e) *(6 points)* Assume $T=\infty$, using a Grimm or Trigger strategy find the minimal δ such that there is a subgame perfect equilibrium where players expect to play (X,χ) in every period. *Be careful to prove your result in every subgame.*

Solution 14 *For the game:*

		Player 2 α	β	ε
Player 1	A	$a+2b; b-c$	$-b; b-c$	$a; b$
	B	$a+b; b+c$	$-b; c-a$	$a-c; b$
	C	$a-b; -b$	$0; 0$	$a+2b; -c$

Let a_t *be the actions that actually occurred in period* t*, then the strategy can be written as:*

$$s_t = \begin{cases} (A,\varepsilon) & \text{if} \quad a_{t-1} = (A,\varepsilon) \\ (C,\beta) & \text{else} \end{cases}$$

First of all, in the subgame where (C,β) *is expected forever we note that no player can change the future payoffs by changing their action today, and that their action today is the NE of the stage game thus it is a best response for both players.*

Secondly, we can also easily dispense with player 2 in the subgame where (A,ε) *is expected forever. If player 2 does the wrong thing today it will decrease her future payoffs and also decrease her current payoffs (since she is supposed to best respond to B) thus she will follow the strategy. This leaves us only needing to check P1 in the* *subgame where (B,β) is always expected.*

$$\begin{aligned} V_1^*(B,\beta) &= u_1(A,\varepsilon)+\frac{\delta}{1-\delta}u_1(A,\varepsilon) \\ &= a+\frac{\delta}{1-\delta}a \\ \hat{V}_1(B,\beta) &= u_1(BR_1(\varepsilon),\varepsilon)+\frac{\delta}{1-\delta}u_1(C,\beta) \\ &= a+2b+\frac{\delta}{1-\delta}0 \end{aligned}$$

$$\begin{aligned} V_1^*(B,\beta) &\geq \hat{V}_1(B,\beta) \\ a+\frac{\delta}{1-\delta}a &\geq a+2b \\ \frac{\delta}{1-\delta}a &\geq 2b \\ \delta a &\geq (1-\delta)2b \\ \delta &\geq \frac{2b}{a+2b}=\delta^* \end{aligned}$$

thus the minimal δ is $\frac{2b}{a+2b}$.

(f) *(3 points)* Note that in the above strategy Player 1 is the only person that has an incentive problem. Rewrite the strategy so that it is only a function of Player 1's actions and prove that this is never an equilibrium.

Solution 15 *The rewritten strategy would be:*

$$s_t=\begin{cases}(A,\varepsilon) & \text{if} \quad a_{1,t-1}=A \\ (C,\beta) & \text{else}\end{cases}$$

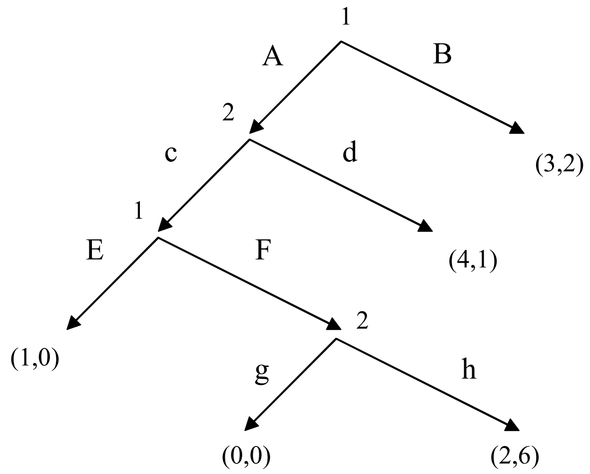
The critical problem with this is that in the subgame (C,β) player 1 can now deviate to A and return to cooperation. So we need to check that they will not deviate or:

$$\begin{aligned} V_1^*(C,\beta) &= 0 \\ \hat{V}_1(C,\beta) &= u_1(A,\beta)+\frac{\delta}{1-\delta}u_1(A,\varepsilon) \\ &= -b+\frac{\delta}{1-\delta}a \end{aligned}$$

$$\begin{aligned} V_1^*(C,\beta) &\geq \hat{V}_1(C,\beta) \\ 0 &\geq -b+\frac{\delta}{1-\delta}a \\ 0 &= -b+\frac{\delta}{1-\delta}a \\ \bar{\delta} &= \frac{b}{a+b}\geq\delta \end{aligned}$$

Now for this strategy to work we must have $\delta \leq \frac{b}{a+b}$ and for the strategy to work when (B, β) is expected we need $\delta \geq \frac{2b}{a+2b}$ but since $\frac{b}{a+b} < \frac{2b}{a+2b}$ this strategy will never work.

4. *(25 points total)* Consider the following sequential game (extensive form game of perfect information).



(a) *(4 points)* Find all the strategies of both players.

Solution 16

$$\begin{aligned} S_1 &= \{A, B\} \times \{E, F\} = \left\{ \begin{array}{c} (A, E), (A, F), \\ (B, E), (B, F) \end{array} \right\} \\ S_2 &= \{c, d\} \times \{g, h\} = \left\{ \begin{array}{c} (c, g), (c, h), \\ (d, g), (d, h) \end{array} \right\} \end{aligned}$$

(b) *(6 points)* Find the subgame perfect or backward induction equilibrium *strategies* and write them down below.

Solution 17 $BR_2(A, c, F) = h$, $BR_1(A, c) = F$, $BR_2(A) = c$, $BR_1(\emptyset) = B$.

$$s_1^* = (B, F), \; s_2^* = (c, h)$$

(c) *(3 points)* Using this game explain why writing down the tactics (the actions that are taken on the equilibrium path) or the outcome (the utilities of both players in equilibrium) are not sufficient.

Solution 18 *If we just write down B or $(3, 2)$ what we have written does not tell us that this is an equilibrium. If player 2 chooses d then B is no longer player 1's best response. We need to write down the entire strategies so that people can see that what we propose is optimal.*

(d) *(6 points)* Using the table below, create a Normal form game that is strategically equivalent to this extensive form game. (There are more rows and columns than necessary.)

		Player 2			
		(c,g)	(c,h)	(d,g)	(d,h)
	(A,E)	$(1,0)$	$(1,0)$	$(4,1)^{12}$	$(4,1)^{12}$
	(A,F)	$(0,0)$	$(2,6)^{2}$	$(4,1)^{1}$	$(4,1)^{1}$
Player 1	(B,E)	$(3,2)^{12}$	$(3,2)^{12}$	$(3,2)^{2}$	$(3,2)^{2}$
	(B,F)	$(3,2)^{12}$	$(3,2)^{12}$	$(3,2)^{2}$	$(3,2)^{2}$

(e) *(6 points)* Find the all of the best responses and pure strategy Nash equilibria of this game. For those Nash equilibria that are not subgame perfect equilibria, explain who makes an empty threat and discuss whether this empty threat benefits them or not.

Solution 19 $NE = \left\{ \begin{array}{c} [(B,E),(c,g)],[(B,E),(c,h)], \\ [(B,F),(c,g)],[(B,F),(c,h)], \\ [(A,E),(d,g)],[(A,E),(d,h)] \end{array} \right\}$ *In the equilibria* $\{[(B,E),(c,g)],[(B,E),(c,h)],[(B,F),(c,g)]\}$ *the only "empty threat" is after the outcome of B has been reached, thus these empty threats neither help nor hurt the person making them. In* $[(A,E),(d,g)],[(A,E),(d,h)]$ *someone makes an empty threat that affects the outcome of the game. You could argue that either P1 or P2 is making the empty threat in some of these equilibria. In both equilibria P1 is playing E and this could be called the empty threat. In which case the empty threat helps player 1. However why is he playing E? In* $[(A,E),(d,h)]$ *it makes no sense, but in* $[(A,E),(d,g)]$ *it is a best response to g, in which case you could say that g is the empty threat—and it actually hurts P2.*