ECON 439
Midterm: Normal Form Games

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This exam will start at 13:40 and finish at 15:20.

Answer all questions in the space provided. Points will only be given for work shown.

1. *(20 points)* Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test.

   Name and Surname: _

   Student ID: _

   Signature:

   _

   χ 2 3 4 5

2. *(16 points total)* Consider a Bertrand Duopoly where each firm chooses $p_i = k_i\delta$ where k_i is a natural number $k_i \in (0, 1, 2, 3, ...)$. The firms have a joint demand curve of $D(P)$ which is downward sloping. Firm i's demand is:

$$d_i(p_i, p_j) = \begin{cases} 0 & \text{if} \quad p_i > p_j \\ \frac{1}{2}D(p_i) & \text{if} \quad p_i = p_j \\ D(p_i) & \text{if} \quad p_i < p_j \end{cases}.$$

   Firm 2 has the marginal cost of χ, but Firm 1 has a marginal cost of 0 with probability $q \in (0, 1)$ and $2 * \chi$ with probability $1 - q \in (0, 1)$. To be clear:

$$\begin{aligned} c_2(q_2) &= \chi q_2 \\ c_1(q_1) &= \begin{cases} 0 & \text{with probability} \quad q \\ 2 * \chi q_2 & \text{with probability} \quad 1 - q \end{cases} \end{aligned}$$

   Assume the δ is "very small" and that the monopoly price is high enough to always be irrelevant.

   (a) *(2 points)* Set up a firm's objective function assuming their marginal cost is c.

   Solution

$$\max_{p_i} (p_i - c)\, d_i(p_i, p_j)$$

   (b) *(4 points)* Find this firm's best response to the other firm's price for all prices less than the monopoly price of this firm. Be very careful about the case when $p_j < c$.

Solution

$$BR_i(p_j) = \begin{cases} p_j - \delta & \text{if } p_j > c + \delta \\ p_j & \text{if } p_j = c + \delta \\ [c, \infty) & \text{if } p_j = c \\ [p_j + \delta, \infty) & \text{if } p_j < c \end{cases}$$

If the price is above marginal cost (plus δ) then by undercutting the price of the opponent one gets on the order of twice the profit, thus this is always a strict best response. If $p_j = c + \delta$ then by raising or lowering the price one gets zero profits, thus one will match the price. If $p_j = c$ lowering the price would result in negative profits, but any price that is the same or higher than the opponent's is fine. If $p_j < c$ all one needs to be certain of is that this firm's price is higher than the opponent's. Thus they are willing to charge any price that is strictly higher, or $[p_j + \delta, \infty)$.

(c) *(4 points)* If firm 2 knows that firm 1's marginal cost is zero, find the set of Nash Equilibria.

Solution If $p_2 = \delta$ then firm one will want to match it, but firm 2 does not want to sell at a price of δ. If $p_2 \geq 2\delta$ then firm one will gladly charge $p_2 - \delta$ and thus firm two will sell nothing. However if $p_2 > \chi$ firm two will not be happy with selling nothing, but in order to undercut firm one they have to charge $p_2 - 2\delta$, If this is weakly less than χ this is fine. So the set of equilibria is:

$$\begin{aligned} p_2 &\in [2\delta, \chi + 2\delta] \\ p_1 &= p_2 - \delta \end{aligned}$$

(d) *(1 point)* If firm 2 knows that firm 1's marginal cost is $2 * \chi$, find the set of Nash Equilibria.

Solution A symmetric argument to the one above establishes that:

$$\begin{aligned} p_1 &\in [\chi + 2\delta, 2\chi + 2\delta] \\ p_2 &= p_1 - \delta \end{aligned}$$

(e) *(5 points)* Find the set of Nash equilibria of the original game, comment.

Solution If $p_2 < \chi$ then now with a strictly positive probability they will loose money. On the other hand if $p_1(0) > \chi + 2\delta$ then firm one will want to undercut this firm and make more profit. But obviously $p_1(0) = p_2 - \chi$ because they have a strictly lower marginal cost. Thus we now only have three equilibria:

$$\begin{aligned} p_2 &\in \{\chi, \chi + \delta, \chi + 2\delta\} \\ p_1(2\chi) &= p_2 + \delta \\ p_1(0) &= p_2 - \delta \end{aligned}$$

This is absolutely fascinating. Notice first of all that—no matter how small q—the "sensible idea" that one should not price below marginal cost is now the law. Second now the type $mc_1 = 0$ firm is now doing much better. The only equilibria that survive is the three best for this firm. Finally, we went from essentially a continuum of equilibria to three—and in the continuum all three would be one equilibrium where $p_2 = \chi$ and $p_1(2\chi)" > "\chi$ and $p_1(0)" < "\chi$. (Those strict inequalities in the continuum means that both prices are the same, but all consumers by from firm 2 if $mc_1 = 2\chi$ and from firm one if $mc_1 = 0$).

a	c	b	$q_i(Q_{-i})$	$q(n)$	$Q(n)$
45	1	4	$\frac{11}{2} - \frac{1}{2}Q_{-i}$	$\frac{11}{n+1}$	$11\frac{n}{n+1}$
45	3	3	$7 - \frac{1}{2}Q_{-i}$	$\frac{14}{n+1}$	$14\frac{n}{n+1}$
40	6	2	$\frac{17}{2} - \frac{1}{2}Q_{-i}$	$\frac{17}{n+1}$	$17\frac{n}{n+1}$
40	8	2	$8 - \frac{1}{2}Q_{-i}$	$\frac{16}{n+1}$	$16\frac{n}{n+1}$

3. *(18 points total)* Consider a symmetric Cournot oligopoly with n firms. Firm choose quantity, $q_i \geq 0$. The price is set based on total quantity, $Q = \sum_{i=1}^{n} q_i$: $P = a - bQ$ and all firms have the same constant marginal cost, $c_i(q) = cq_i$; let $Q_{-i} = \sum_{j \neq i} q_j$.

   (a) *(2 points)* Find the firm's objective function using only q_i and Q_{-i} or Q.

   $$\max_{q_i} (a - b(Q_{-i} + q_i))\, q_i - cq_i$$

   (b) *(4 points)* Find the firm's first order condition and best response.

   $$\begin{aligned} (a - b(Q_{-i} + q_i)) - bq_i - c &= 0 \\ q_i &= \frac{1}{2}\frac{a-c}{b} - \frac{1}{2}Q_{-i} \end{aligned}$$

   At this point it is useful to write $\frac{a-c}{b} = Q_\infty$, *and I will do this throughout.*

   (c) *(3 points)* Using symmetry, find a Nash equilibrium. (All answers from this point on will be a function of n.)

   Solution Now we know that $Q^* = nq^*$ and working with the FOC:

   $$\begin{aligned} a - bnq - bq - c &= 0 \\ q^* &= \frac{1}{n+1}Q_\infty \\ Q^* &= \frac{n}{n+1}Q_\infty \end{aligned}$$

Working with the best response $Q_{-i} = (n-1)\,q^*$ and

$$\begin{aligned} q &= \frac{1}{2}Q_\infty - \frac{(n-1)}{2}q \\ 2q &= Q_\infty - (n-1)\,q \\ (n+1)\,q &= Q_\infty \\ q^* &= \frac{1}{n+1}Q_\infty \end{aligned}$$

(d) *(5 points)* Find the unique Nash equilibrium **without** using symmetry. (**Hint:** First find a way to solve for the equilibrium market quantity Q^*, and then find q_i^*.)

Solution The trick here is to sum the first order conditions.

$$\begin{aligned} a - bQ^* - bq_i^* - c &= 0 \\ (a - bQ^* - c) - bq_i^* &= 0 \end{aligned}$$

The term $(a - bQ^* - c)$ appears in every first order condition, or n times. The last term will only appear once. So the sum is:

$$n\,(a - bQ^* - c) - b\sum_{i=1}^{n} q_i^* = 0$$

But of course $Q^* = \sum_{i=1}^{n} q_i^*$ thus we have

$$\begin{aligned} n\,(a - bQ^* - c) - bQ^* &= 0 \\ n\,(a-c) &= (n+1)\,bQ^* \\ Q^* &= \frac{n}{n+1}Q_\infty \end{aligned}$$

taking this back to the first order conditions we recognize that

$$\begin{aligned} a - b\left(\frac{n}{n+1}Q_\infty\right) - bq_i - c &= 0 \\ \frac{1}{n+1}\,(a - c - bq_i - bnq_i) &= 0 \\ q_i^* &= \frac{1}{n+1}Q_\infty \end{aligned}$$

which is exactly what we had before.
It is harder to do with the best responses, but equally possible After the summation we would have:Working with the best response $Q_{-i} = (n-1)\,q^*$ and

$$Q^* = n\frac{1}{2}Q_\infty - \frac{1}{2}\sum_{i=1}^{n} Q_{-i}^*$$

and we need to think about the fact that in $\sum_{i=1}^{n} Q^*_{-i}$ q^*_j will appear exactly $n-1$ times for every j. Thus

$$\sum_{i=1}^{n} Q^*_{-i} = (n-1)\, Q^*$$

$$Q^* = n\frac{1}{2}Q_\infty - \frac{1}{2}(n-1)\, Q^*$$

and the rest is just some algebra. Notice we really need to go to the FOC to find q^*_i.
I found another method proposed on one of the exams. One notices that $Q_{-i} = Q - q_i$ then:

$$\begin{aligned} q_i &= \frac{1}{2}Q_\infty - \frac{1}{2}(Q^* - q_i) \\ q_i &= \frac{1}{2}r - \frac{1}{2}Q^* + \frac{1}{2}q_i \\ q_i &= Q_\infty - Q^* \end{aligned}$$

Now at this point we sum over i and get:

$$\begin{aligned} Q &= n\,(Q_\infty - Q) \\ Q &= \frac{n}{n+1}Q_\infty \end{aligned}$$

and going back to the best response we see immediately:

$$q_i = Q_\infty - \frac{n}{n+1}Q_\infty = \frac{1}{n+1}Q_\infty$$

a very nice method.

(e) *(2 points)* Find the set of dominated strategies in this game.

Solution In this game $q_i \in [0, \infty)$ and

$$\begin{aligned} q_i &= \frac{1}{2}Q_\infty - \frac{1}{2}Q_{-i} \\ \frac{\partial q_i}{\partial Q_{-i}} &= -\frac{1}{2} < 0 \end{aligned}$$

Thus the maximum value for q_i will occur when $Q_{-i} = 0$, and the minimum will occur when $Q_{-i} \to \infty$. The former gives us:

$$q_i \leq \frac{1}{2}Q_\infty$$

and the latter gives us

$$q_i \geq \max\left\{0, \lim_{Q_{-i}\to\infty}\left(\frac{1}{2}Q_\infty - \frac{1}{2}Q_{-i}\right)\right\} = 0$$

Thus the dominated strategies in this game are $q_i > \frac{1}{2}Q_\infty$.

(f) *(5 points)* Prove that if $n > 2$ then this equilibrium can not be found by using iterated deletion of dominated strategies by showing that there is no $\underline{q}$ and $\overline{q}$ ($\underline{q} < \overline{q}$) such that if $q_j \in [\underline{q}, \overline{q}]$ for all $j \neq i$ then the best response of i is in the interior of $(\underline{q}, \overline{q})$.

Solution I really should have been more precise here, the statement is fine, but one could (without loss of generality) include $0 < \underline{q} < q^*(n) < \overline{q} < \frac{1}{2}Q_\infty$ and that would have made it a little simpler. In fact that is what I will be proving, though strict inequalities below could be replaced with weak ones without loss of generality.

The best response to everyone playing $\overline{q}$ is

$$q(\overline{q}) = \frac{1}{2}Q_\infty - \frac{1}{2}(n-1)\overline{q}$$

and for $\underline{q}$ it is

$$q(\underline{q}) = \frac{1}{2}Q_\infty - \frac{1}{2}(n-1)\underline{q}$$

In order for the statement to be true we need:

$$\begin{aligned} \underline{q} &> \frac{1}{2}Q_\infty - \frac{1}{2}(n-1)\overline{q} \\ \overline{q} &> \frac{1}{2}Q_\infty - \frac{1}{2}(n-1)\underline{q} \end{aligned}$$

combining these statements we get:

$$\begin{aligned} \overline{q} &> \frac{1}{2}Q_\infty - \frac{1}{2}(n-1)\left(\frac{1}{2}Q_\infty - \frac{1}{2}(n-1)\overline{q}\right) \\ \overline{q} &> -\frac{1}{4b}(a-c)(n-3) + \frac{1}{4}\overline{q}(n-1)^2 \\ -\frac{1}{4b}(a-c)(n-3) &< \frac{1}{4}\overline{q}(n-1)^2 - \overline{q} \\ \frac{1}{4b}(a-c)(n-3) &< \frac{1}{4}\overline{q}\left(n^2-2n-3\right) \\ \frac{\frac{1}{4b}(a-c)(n-3)}{\frac{1}{4}(n^2-2n-3)} &< \overline{q} \\ \frac{1}{n+1}Q_\infty &< \overline{q} \end{aligned}$$

and thus this will be true for any $\overline{q} > q^* = \frac{1}{n+1}Q_\infty$.

4. *(36 points)* Consider the following normal form game, where $\lambda > a$.

Solution Explanation of notation on the games. $[t]$ for $t \in \{1, 2, 3, 4\}$ means that I removed a given strategy in the t'th round of deletion of dominated strategies. $(q(\lambda), q(\lambda^*))$ are the probability of that strategy in the unique mixed strategy equilibrium as a function of λ, then as a function of λ^*, the rest of the weight goes on the other strategy in the mixed strategy Nash equilibrium. I only do this for P2 because for P1 the answer is $\frac{1}{2}$ for both strategies.

Player 2

Player 1	α	β	δ	γ	ε
A	$\lambda; b$	$a+c; b-c$	$b-f; b+d-f$	$d-c; b-f$	$a-c; b+d$
B	$a; b+d$	$c; b-f$	$b; b-c$	$d-b; b-2c$	$a; b$
C	$a-2b; c$	$c-f; -a$	$b-c; b+c$	$d+b; b$	$a-g; b$
D	$a-d; b-f$	$c-d; b-2f$	$b-d; b-2a$	$d-2b; b$	$a-d; b-d$
E	$a-b; a$	$c-d; a$	$b-f; a+c$	$d; a-f$	$a-f; -c$

a	b	c	d	f	g
1	2	3	4	5	6

Player 2

Player 1	α	$\beta\,[1]$	$\delta\,[4]$	$\gamma\,[2]$	$\varepsilon\left(\frac{3}{\lambda+2}, \frac{1}{3}\right)$
A	$\lambda; 2^1$	$4; -1^1$	$-3; 1$	$1; -3$	$-2; 6^2$
B	$1; 6^2$	$3; -3$	$2; -1^1$	$-2; -4$	$1; 2^1$
$C\,[3]$	$-3; 3$	$-2; -1$	$-1; 5^2$	$6; 2^1$	$-5; 2$
$D\,[1]$	$-3; -3$	$-1; -8$	$-2; 0$	$0; 2^2$	$-3; -2$
$E\,[3]$	$-1; 1$	$-1; 1$	$-3; 4^2$	$4; -4$	$-4; -3$

Player 2

Player 1	α	β	δ	γ	ε
A	$a+c; b-c$	$b-f; b+d-f$	$d-c; b-f$	$a-c; b+d$	$\lambda; b$
B	$c; b-f$	$b; b-c$	$d-b; b-2c$	$a; b$	$a; b+d$
C	$c-d; b-2f$	$b-d; b-2a$	$d-2b; b$	$a-d; b-d$	$a-d; b-f$
D	$c-f; -a$	$b-c; b+c$	$d+b; b$	$a-g; b$	$a-2b; c$
E	$c-d; a$	$b-f; a+c$	$d; a-f$	$a-f; -c$	$a-b; a$

a	b	c	d	f	g
6	1	2	3	5	4

Player 2

Player 1	$\alpha\,[4]$	$\beta\,[4]$	$\delta\,[2]$	$\gamma\left(\frac{\lambda-6}{\lambda-4}, \frac{1}{3}\right)$	ε
A	$8; -1^1$	$-4; -1$	$1; -4$	$4; 4^2$	$\lambda; 1^1$
B	$2; -4$	$1; -1^1$	$2; -3$	$6; 1^1$	$6; 4^2$
$C\,[1]$	$-1; -9$	$-2; -11$	$1; 1^2$	$3; -2$	$3; -4$
$D\,[3]$	$-3; -6$	$-1; 3^2$	$4; 1^1$	$-2; 1$	$4; 2$
$E\,[3]$	$-1; 6$	$-4; 8^2$	$3; 1$	$1; -2$	$5; 6$

Player 2

Player 1		α	β	δ	γ	ε
	A	$a-c; b+d$	$a+c; b-c$	$b-f; b+d-f$	$d-c; b-f$	$\lambda; b$
	B	$a; b$	$c; b-f$	$b; b-c$	$d-b; b-2c$	$a; b+d$
	C	$a-d; b-d$	$c-d; b-2f$	$b-d; b-2a$	$d-2b; b$	$a-d; b-f$
	D	$a-f; -c$	$c-d; a$	$b-f; a+c$	$d; a-f$	$a-b; a$
	E	$a-g; b$	$c-f; -a$	$b-c; b+c$	$d+b; b$	$a-2b; c$

a	b	c	d	f	g
5	6	1	2	3	4

Player 2

Player 1		$\alpha\left(\frac{\lambda-5}{\lambda-4}, \frac{1}{5}\right)$	β [4]	δ [4]	γ [2]	ε
	A	$4; 8^2$	$3; 5^1$	$3; 5$	$1; 3$	$\lambda; 6^1$
	B	$5; 6^1$	$1; 3$	$6; 5^1$	$-4; 4$	$5; 8^2$
	C [1]	$3; 4$	$-1; 0$	$4; -4$	$-10; 6^2$	$3; 3$
	D [3]	$2; -1$	$-1; 5$	$3; 6^2$	$2; 2$	$-1; 5$
	E [3]	$1; 6$	$-2; -5$	$5; 7^2$	$8; 6^1$	$-7; 1$

Player 2

Player 1		α	β	δ	γ	ε
	A	$a; b$	$c; b-f$	$d-b; b-2c$	$b; b-c$	$a; b+d$
	B	$a-d; b-d$	$c-d; b-2f$	$d-2b; b$	$b-d; b-2a$	$a-d; b-f$
	C	$a-c; b+d$	$a+c; b-c$	$d-c; b-f$	$b-f; b+d-f$	$\lambda; b$
	D	$a-f; -c$	$c-d; a$	$d; a-f$	$b-f; a+c$	$a-b; a$
	E	$a-g; b$	$c-f; -a$	$d+b; b$	$b-c; b+c$	$a-2b; c$

a	b	c	d	f	g
4	5	6	1	3	2

Player 2

Player 1		$\alpha\left(\frac{\lambda-4}{\lambda+2}, \frac{3}{4}\right)$	β [4]	δ [2]	γ [4]	ε
	A	$4; 5^1$	$6; 2$	$-4; -2$	$5; -1^1$	$4; 6^2$
	B [1]	$3; 4$	$5; -1$	$-9; 5^2$	$4; -3$	$3; 2$
	C	$-2; 6^2$	$10; -1^1$	$-5; 2$	$2; 3$	$\lambda; 5^1$
	D [3]	$1; -6$	$5; 4$	$1; 1$	$2; 10^2$	$-1; 4$
	E [3]	$2; 5$	$3; -4$	$6; 5^1$	$-1; 11^2$	$-6; 6$

(a) *(10 points)* Find all the pure strategy best responses of both players. You may mark them on the game but you will loose two points if you do not explain your notation below.

Solution I will mark my answers on each game by writing a 1 in the upper right hand corner for a BR for player 1, a 2 for player 2.

(b) *(4 points)* Find the unique dominated strategy in this game. State what dominates it and clearly explain why this strategy is dominated, making a payoff by payoff comparison.

Solution I will work with the game:

		Player 2				
		α	β	δ	γ	ε
Player 1	A	$a; b^1$	$c; b-f$	$d-b; b-2c$	$b; b-c^1$	$a; b+d^2$
	B	$a-d; b-d$	$c-d; b-2f$	$d-2b; b^2$	$b-d; b-2a$	$a-d; b-f$
	C	$a-c; b+d^2$	$a+c; b-c^1$	$d-c; b-f$	$b-f; b+d-f$	$\lambda; b^1$
	D	$a-f; -c$	$c-d; a$	$d; a-f$	$b-f; a+c^2$	$a-b; a$
	E	$a-g; b$	$c-f; -a$	$d+b; b^1$	$b-c; b+c^2$	$a-2b; c$

to provide detailed answers, and then provide short hand answers for the other variations.
In this game there are three strategies that are never pure strategy best responses, these are: $\{B, D, \beta\}$. Obviously you only need to find the dominated strategy, but I will proceed by defining the function:

$$BT_i(X, Y) = \{s_i \in S_i | u_i(s_i, X) > u_i(Y, X)\}$$

$$\{A, C, D, E\} \subseteq BT_i(\alpha, B) \subseteq \{A\}$$

we can't be sure if C, D, or E are better, we can be sure that A is.

$$\begin{aligned}
\{A, C, E\} &\subseteq BT_i(\beta, B) \subseteq \{A, C\} \\
\{A, C, D, E\} &\subseteq BT_i(\delta, B) \subseteq \{A, D, E\} \\
\{A, C, D, E\} &\subseteq BT_i(\gamma, B) \subseteq \{A\} \\
\{A, C, D\} &\subseteq BT_i(\varepsilon, B) \subseteq \{A, C\}
\end{aligned}$$

Thus since $A \in \cap_{s_2 \in S_2} BT_i(s_2, B)$ we know that A strictly dominates B. Because:

$$\begin{aligned}
u_1(\alpha, A) &= a > a - d = u_1(\alpha, B), \; u_1(\beta, A) = c > c - d = u_1(\beta, B), \\
u_1(\delta, A) &= d - b > d - 2b = u_1(\delta, B), \; u_1(\gamma, A) = b > b - d = u_1(\gamma, B), \\
u_1(\varepsilon, A) &= a > a - d = u_1(\varepsilon, B)
\end{aligned}$$

At this point you could stop, but of course I have to go on.

$$\begin{aligned}
\{A, B, C, E\} &\subseteq DT_i(\alpha, D) \subseteq \{A\} \\
\{A, B, C, E\} &\subseteq DT_i(\beta, D) \subseteq \{A, C\} \\
DT_i(\delta, D) &= \{E\} \\
\{A, B, C, E\} &\subseteq DT_i(\gamma, D) \subseteq \{A\} \\
\{A, B, C\} &\subseteq DT_i(\varepsilon, D) \subseteq \{A, C\}
\end{aligned}$$

So there is no common element and it is not dominated.

$$\begin{aligned}
\{\alpha,\delta,\gamma,\varepsilon\} &\subseteq BT_i(A,\beta)\subseteq\{\alpha,\varepsilon\}\\
\{\alpha,\delta,\gamma,\varepsilon\} &\subseteq BT_i(B,\beta)\subseteq\{\delta,\varepsilon\}\\
\{\alpha,\delta,\gamma,\varepsilon\} &\subseteq BT_i(C,\beta)\subseteq\{\alpha,\varepsilon\}\\
BT_i(D,\beta) &= \{\gamma\}\\
BT_i(E,\beta) &= \{\alpha,\delta,\gamma,\varepsilon\}
\end{aligned}$$

Thus I find out the question was wrong, it is possible that γ might dominate β for some variations of the quiz. Notice that α is removed against D because it has the same utility, so it's only *weakly dominant.* I hope none of you removed it.

(c) *(10 points)* Using iterated deletion of dominated strategies, find the two strategies that survive for both players. When you remove a strategy state what dominates the given strategy.

Solution I'm going to proceed a little more add hock here. The remaining game (removing the one strategy I can be sure I can remove) is:

		Player 2				
		α	β	δ	γ	ε
	A	$a;b^1$	$c;b-f$	$d-b;b-2c$	$b;b-c^1$	$a;b+d^2$
Player 1	C	$a-c;b+d^2$	$a+c;b-c^1$	$d-c;b-f$	$b-f;b+d-f$	$\lambda;b^1$
	D	$a-f;-c$	$c-d;a$	$d;a-f$	$b-f;a+c^2$	$a-b;a$
	E	$a-g;b$	$c-f;-a$	$d+b;b^1$	$b-c;b+c^2$	$a-2b;c$

Now for P2 $\{\beta,\delta\}$ are never pure strategy best responses. We can analyze β using our previous analysis:

$$\begin{aligned}
\{\alpha,\delta,\gamma,\varepsilon\} &\subseteq BT_i(A,\beta)\subseteq\{\alpha,\varepsilon\}\\
\{\alpha,\delta,\gamma,\varepsilon\} &\subseteq BT_i(C,\beta)\subseteq\{\alpha,\varepsilon\}\\
BT_i(D,\beta) &= \{\gamma\}\\
BT_i(E,\beta) &= \{\alpha,\delta,\gamma,\varepsilon\}
\end{aligned}$$

and it is unchanged. For δ:

$$\begin{aligned}
\{\alpha,\beta,\gamma,\varepsilon\} &\subseteq BT_i(A,\delta)\subseteq\{\alpha,\gamma,\varepsilon\}\\
\{\alpha,\beta,\gamma,\varepsilon\} &\subseteq BT_i(C,\delta)\subseteq\{\alpha,\gamma,\varepsilon\}\\
BT_i(D,\delta) &= \{\beta,\gamma,\varepsilon\}\\
\{\gamma,\varepsilon\} &\subseteq BT_i(E,\delta)\subseteq\{\gamma\}
\end{aligned}$$

so γ dominates δ.

		Player 2			
		α	β	γ	ε
	A	$a; b^1$	$c; b-f$	$b; b-c^1$	$a; b+d^2$
Player 1	C	$a-c; b+d^2$	$a+c; b-c^1$	$b-f; b+d-f$	$\lambda; b^1$
	D	$a-f; -c$	$c-d; a$	$b-f; a+c^2$	$a-b; a$
	E	$a-g; b$	$c-f; -a$	$b-c; b+c^2$	$a-2b; c$

Now only $\{A, C\}$ are BR for P1, and one can quickly see that A dominated $\{D, E\}$

		Player 2			
		α	β	γ	ε
	A	$a; b^1$	$c; b-f$	$b; b-c^1$	$a; b+d^2$
Player 1	C	$a-c; b+d^2$	$a+c; b-c^1$	$b-f; b+d-f$	$\lambda; b^1$

Now only $\{\alpha, \varepsilon\}$ are BR for P2, and α dominates β and γ.

(d) *(4 points)* Write down the remaining game in the table provided below and find the unique Nash equilibrium of this game. The equilibrium will be a function of λ.

		Player 2	
		$\alpha\,(q)$	$\varepsilon\,(1-q)$
	$A\,(p)$	$a; b^1$	$a; b+d^2$
Player 1	$C\,(1-p)$	$a-c; b+d^2$	$\lambda; b^1$

$$
\begin{aligned}
U_1(A, q) &= qa + (1-q)\,a = a \\
U_1(C, q) &= q\,(a-c) + (1-q)\,\lambda = \lambda - q\lambda + aq - cq \\
\lambda - q\lambda + aq - cq &= a \\
q &= \frac{\lambda - a}{\lambda - a + c}
\end{aligned}
$$

$$
\begin{aligned}
U_2(\alpha, p) &= pb + (1-p)\,(b+d) \\
U_2(\varepsilon, p) &= p\,(b+d) + (1-p)\,b \\
pb + (1-p)\,(b+d) &= p\,(b+d) + (1-p)\,b \\
p &= \frac{1}{2}
\end{aligned}
$$

(e) *(8 points total)* Now assume that this is a Bayesian game, with λ distributed uniformly over $[a, a+2g]$ or that $\Pr(\lambda \leq x) = F(x) = \frac{x-a}{2g}$.

i. *(2 points)* Argue now that Player 1 will use a cut off strategy, I.e. will play one action of $\lambda \geq \lambda^*$ and another if $\lambda \leq \lambda^*$. Specify this strategy.

Solution If λ is high this increases the value of C, thus sensibly if my λ is high I should play C, and if it's low I should play A. Thus the strategy is:

$$S_1(\lambda) = \begin{cases} C & \text{if} \quad \lambda \geq \lambda^* \\ A & \text{if} \quad \lambda \leq \lambda^* \end{cases}$$

ii. *(2 points)* Find a formula for λ^* as a function of the mixed strategy of player 2.

Solution Given q, we must have:

$$\begin{aligned} q(a-c) + (1-q)\lambda^* &= a \\ \frac{1}{1-q}(a(1-q)+cq) &= \lambda^* \end{aligned}$$

iii. *(4 points)* Find the equilibrium of this game.

Solution In this new game $p = \Pr(A) = F(\lambda^*)$ so we have

$$\begin{aligned} \frac{1}{2} &= \frac{\lambda^* - a}{2g} \\ \lambda^* &= a + g \end{aligned}$$

Thus from the formula for λ^* above we have

$$\begin{aligned} \frac{1}{1-q}(a(1-q)+cq) &= a+g \\ q &= \frac{g}{c+g} \end{aligned}$$

5. *(10 points total)* We know that one of two events has occurred, X or Y, with $\Pr(X) = \pi \in (0,1)$. We observe one or more independent signals. The signals are x or y, with $\Pr(x|X) = p \in \left(\frac{1}{2}, 1\right)$ and $\Pr(y|Y) = q \in \left(\frac{1}{2}, 1\right)$.

(a) *(5 points)* Assume we have observed one signal and it was x. What is the probability that the true state is X?

$$\begin{aligned} \Pr(A|B) &= \frac{\Pr(A \cap B)}{\Pr B} \\ \Pr(A|B)\Pr B &= \Pr(A \cap B) \\ \Pr(A|B) &= \frac{\Pr(B|A)\Pr A}{\Pr B} \end{aligned}$$

here B is x and A is X.

$$\begin{aligned} \Pr(X|x) &= \frac{\Pr(x|X)\Pr X}{\Pr(x|X)\Pr X + \Pr(x|Y)\Pr Y} \\ &= \frac{p\pi}{p\pi + (1-q)(1-\pi)} \end{aligned}$$

(b) *(3 points)* Assume we have observed two signals and both were x. What is the probability that the true state is X? Show it is higher than before.

$$\Pr(X|x,x) = \frac{\Pr(x,x|X)\Pr X}{\Pr(x,x|X)\Pr X + \Pr(x,x|Y)\Pr Y}$$

$$\Pr(x,x|X) = \Pr(x|X)\Pr(x|X)$$

by independence.

$$\Pr(X|x,x) = \frac{p^2\pi}{p^2\pi + (1-q)^2(1-\pi)}$$

What we need to show is that

$$\begin{aligned}
\frac{p^2\pi}{p^2\pi + (1-q)^2(1-\pi)} &> \frac{p\pi}{p\pi + (1-q)(1-\pi)} \\
p^2\pi\left(p\pi + (1-q)(1-\pi)\right) &> p\pi\left(p^2\pi + (1-q)^2(1-\pi)\right) \\
\frac{p^2\pi}{p\pi}\left(p\pi + (1-q)(1-\pi)\right) &> p^2\pi + (1-q)^2(1-\pi) \\
p + \pi p^2 - \pi p - pq + \pi pq &> p^2\pi + (1-q)^2(1-\pi) \\
p + \pi p^2 - \pi p - pq + \pi pq - \left(p^2\pi + (1-q)^2(1-\pi)\right) &= (1-\pi)(1-q)(p+q-1) > 0
\end{aligned}$$

and this is strictly positive because $p+q>1$ of course the way I did it in class was:

$$\Pr(X|x,x) = \frac{p\pi}{p\pi + (1-q)(1-\pi)\frac{1-q}{p}}$$

and since $\frac{1-q}{p} < 1$ the second expression must be strictly higher (numerator same, denominator smaller.)

(c) *(2 points)* Explain why this model is fundamental tool for understanding the implications of data.

Solution Our primary interest is the underlying true state, however our data will be generated *conditional* on that state, like the signals here. We thus need to invert our observations, like we did in this exercise, in order to find out the probability of the true state. Since $p > \frac{1}{2}$ and $q > \frac{1}{2}$ with enough observations (which will not always be either x or y) we will be able to pick out the true state from the data using this sort of reasoning.