ECON 439
Midterm: Extensive Form Games

Kevin Hasker

This exam will start at 18:40 and finish at 20:20.

Answer all questions in the space provided. Points will only be given for work shown.

1. *(16 points)* Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test.

   Name and Surname: _

   Student ID: _

   Signature:

   _

2. *(23 points total)* Consider and infinitely repeated game where payoffs t periods in the future are discounted by δ^{t-1}, for $\delta \in (0,1)$. The stage game is:

	L	C	R
U	$a+b; c+b$	$-3; -a$	$-c; c+b+1$
M	$a+b+1; c-3$	$-1; c-2$	$a; c$
D	$a; -b$	$0; 0$	$a+3; -c$

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	L	C	R
U	$13; 10$	$-3; -5$	$-2; 11$
M	$14; -1$	$-1; 0$	$5; 2$
D	$5; -8$	$0; 0$	$8; -2$

X	Y	p	q	a	b	c
M	R	D	C	5	8	2

	L	C	R
U	$a+b+1; c-3$	$-1; c-2$	$a; c$
M	$a; -b$	$0; 0$	$a+3; -c$
D	$a+b; c+b$	$-3; -a$	$-c; c+b+1$

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	L	C	R
U	$8; -1$	$-1; 0$	$4; 2$
M	$4; -3$	$0; 0$	$7; -2$
D	$7; 5$	$-3; -4$	$-2; 6$

X	Y	p	q	a	b	c
U	R	M	C	4	3	2

	L	C	R
U	$a+b+1; c-3$	$a; c$	$-1; c-2$
M	$a; -b$	$a+3; -c$	$0; 0$
F	$a+b; c+b$	$-c; c+b+1$	$-3; -a$

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	L	C	R
U	$11; -2$	$3; 1$	$-1; -1$
M	$3; -7$	$6; -1$	$0; 0$
F	$10; 8$	$-1; 9$	$-3; -3$

X	Y	p	q	a	b	c
U	C	M	R	3	7	1

	L	C	R
U	$a; -b$	$a+3; -c$	$0; 0$
M	$a+b; c+b$	$-c; c+b+1$	$-3; -a$
D	$a+b+1; c-3$	$a; c$	$-1; c-2$

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	L	C	R
U	$5; -2$	$8; -3$	$0; 0$
M	$7; 5$	$-3; 6$	$-3; -5$
D	$8; 0$	$5; 3$	$-1; 1$

X	Y	p	q	a	b	c
D	C	U	R	5	2	3

(a) *(3 points)* Find the pure strategy best responses for both players in the stage game. You may mark them on the table above, but you will loose a point if you do not explain your notation below.

(b) *(2 points)* Find the unique Nash equilibrium of this stage game. Explain why it is a Nash equilibrium.

(c) *(4 points)* Define a minimax strategy and find it in this stage game for both players.

(d) *(3 points)* Find a trigger or Grimm strategy that can support (X, Y) as the initial path of a subgame perfect equilibrium.

$$s_t = \begin{cases} (X, Y) & \text{if } (X, Y) \text{ occured last period} \\ (p, q) & \text{else} \end{cases}$$

(e) *(5 points)* Prove that this is a subgame perfect equilibrium in all subgames, and find the minimal δ such that it is an equilibrium.

First the simple subgame, where we are going to play (p, q) today. Since neither player alone can affect the future and the current strategy is a Nash equilibrium this is an equilibrium.

Next, the subgame where we expect (X, Y). We do not need to analyze player two because she is both playing a best response and doing anything else will make the future worst. Thus we only need to analyze P1, and:

$$\begin{aligned} V_1^*(X, Y) &= a + \frac{\delta}{1-\delta} a \\ V_1'(X, Y) &= a + 3 + \frac{\delta}{1-\delta} 0 \end{aligned}$$

$$\begin{aligned} V_1^*(X, Y) - V_1'(X, Y) &= \left(a + \frac{\delta}{1-\delta} a\right) - \left(a + 3 + \frac{\delta}{1-\delta} 0\right) \\ &= \frac{1}{1-\delta}(3\delta + a\delta - 3) \geq 0 \end{aligned}$$

$$\begin{aligned} 3\delta + a\delta - 3 &\geq 0 \\ \delta &\geq \frac{3}{a+3} \end{aligned}$$

(f) *(3 points)* Prove that the following strategy is not a subgame perfect equilibrium for any δ.

$$s_t = \begin{cases} (X,Y) & \text{if Player 1 played } X \text{ in the last period} \\ (p,q) & \text{else} \end{cases}$$

Above we established that this strategy would be an equilibrium when the state was (X,Y) *if* $\delta \geq \frac{3}{a+3}$. *With this strategy player 1 can now "rebuild his reputation" by playing* X *instead of* p. *If he does this he will get:*

$$V_1'(p,q) = -1 + \frac{\delta}{1-\delta}a$$

and in order for this to be an equilibrium we must have:

$$\begin{aligned} V_1^*(p,q) &= 0 \geq -1 + \frac{\delta}{1-\delta}a \\ \delta &\leq \frac{1}{a+1} \end{aligned}$$

thus there is no δ *where players will actually follow this strategy.*

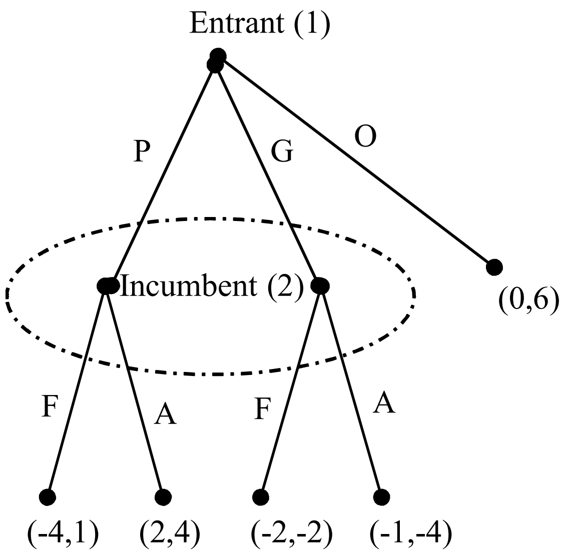
(g) *(3 points)* For a correlated strategy c, what can we conclude about $(u_1(c), u_2(c))$ in order for it to be the initial path of a subgame perfect equilibrium for some δ? Explain your answer.

We must have $u_i(c) > 0$ *for both players. We could use the trigger strategy above to prove the conclusion for any* c, *and since* $(0,0)$ *is the (pure strategy) minmax of this game this is as low as we can go.*

3. *(3 points)* Krazy Ken and Straight Stu both grade on the curve. Stu, like all reasonable professors, adjusts the grades at the end of the semester so that the average grade is what Stu wants. Ken adjusts the grades after each assignment so that those who completed the assignment get the average grade Ken wants. Ken has noticed that a higher percentage of students are showing up for his exams than Stu's. Assuming students are rational and want to maximize their grade explain this phenomena.

   Consider a hard assignment, one where most people are probably going to get below 30% of the possible points. Under Straight Stu's grading scheme, someone who doesn't complete the assignment will get only 30% fewer points than the best in the class. Under Krazy Ken's grading scheme this same student would basically loose the average grade in the class, probably something in the 70% to 80% range. Thus completing assignments is much more important in Krazy Ken's class, and this should carry over to exams.

4. *(16 points total)* Consider an entry game where the entrant (P1) now has the choice of being aggressive. The incumbent (P2) does not know if the entrant is aggressive (G) or passive (P). The extensive form game is:



(a) *(5 points)* Solve this game as if the incumbent knows whether the Entrant chooses P or G. Is this a strategy of the game as specified? Is it a weak sequential equilibrium? (You may indicate your best response on the game above, but you will loose two points if you do not explain your notation.)

(b) *(6 points)* Find the other weak sequential equilibrium of this game, be sure to specify all beliefs with intervals if possible.

(c) *(2 points)* Notice that G is a strictly dominated strategy, does this mean that it can't affect the set of weak sequential equilibria? Why or why not?

(d) *(3 points)* In this game there would be a benefit to the entrant of revealing her plans to the incumbent, or letting the incumbent know whether she chose P or G. Would this benefit occur in an arbitrary extensive form game? Why or why not? (We will continue to assume sequential rationality in this new game.)

a	b	k_1	k_2	c_1	c_2	$\underline{t}$	$\bar{t}_1$	$\bar{t}_2$	$\pi_i^{\bar{t}_j+1+}$	$\underline{\pi}_i^{\bar{t}_j}$	$\pi_i^{\bar{t}_j+}$
32	$\frac{1}{2}$	14	20	9	2	6	16	20	120	-60	60
60	2	4	12	24	4	4	28	32	72	-48	24
30	$\frac{1}{2}$	20	16	4	10	2	16	12	120	-80	40
66	3	9	3	3	25	5	36	32	54	-45	9

5. *(20 points total)* Consider an industry which has the demand curve $P(t, Q) = a - t - bQ$, where $t \in \{0, 1, 2, ...\}$ is the time period and Q is the total output. There are two firms in this industry, their strategy is simply the

last period in which they produce—$t_i \in \{0, 1, 2, ...\}$. In period t, if $t \leq t_i$ they produce k_i ($k_1 = \alpha$, $k_2 = \beta$) if $t > t_i$ they produce no output. They have constant marginal cost, and their total costs are $c_1(q_1) = \delta q_1$, $c_2(q_2) = \gamma q_2$. Their profit in period t is $\pi_{it} = (P(t, Q_t) - c_i) q_{it}$ where $q_{it} = 0$ if $t > t_i$ and k_i if $t \leq t_i$. Their objective is to maximize the sum of their profits in all periods.

Assume a firm will choose to produce if indifferent. And please notice that for simplicity price can be negative.

(a) *(6 points)* Let $\underline{t}$ be the last period in which if both firms produce they will make at least zero profit, $\bar{t}_i$ be the last period in which firm i produces alone then they can make at least zero profit. Find $(\underline{t}, \bar{t}_1, \bar{t}_2)$ note that $\bar{t}_2 \neq \bar{t}_1 > \underline{t}$.

What matters is when the price minus marginal cost is positive. Thus we need to know when:

$$\begin{aligned} 0 &= P(t, k_1 + k_2) - \max(c_1, c_2) \\ 0 &= a - \underline{t} - b(\alpha + \beta) - \max(\delta, \gamma) \\ \underline{t} &= a - b(\alpha + \beta) - \max(\delta, \gamma) \end{aligned}$$

$$\begin{aligned} 0 &= a - t - b\alpha - c_1 \\ \bar{t}_1 &= a - \delta - b\alpha \\ \bar{t}_2 &= a - \gamma - b\beta \end{aligned}$$

(b) *(4 points)* Prove that $t_i \in \{\underline{t}, \underline{t}+1, \underline{t}+2, ..., \bar{t}_i\}$ for both i.

Solution 1 *If $t_i < \underline{t}$ then a firm can make a weakly positive profit in the periods between t_i and $\underline{t}$, thus they will produce by assumption. If $t_i > \bar{t}_i$ they must be strictly losing money between $\bar{t}_i$ and t_i, thus they should choose at least $\bar{t}_i$.*

(c) *(3 points)* Assuming that $\min\{t_1, t_2\} \geq \min\{\bar{t}_1, \bar{t}_2\} - 1$, find the unique equilibrium values of (t_1, t_2).

Solution 2 *Assume $\bar{t}_1 < \bar{t}_2$, if the inequality is reversed for your exam simply reverse the answers below. If firm 2 choose $t_2 \geq \bar{t}_1$ then they should choose $t_2 = \bar{t}_2$. In the periods $\bar{t}_1 + 1$ to $\bar{t}_2$ they will make:*

$$\begin{aligned} \pi_2^{\bar{t}_1+1+} &= \sum_{t=\bar{t}_1+1}^{\bar{t}_2} (a - t - b\beta - \gamma)\beta \\ &= \beta \left(\sum_{t=a-\delta-b\alpha+1}^{(a-\gamma-b\beta)} (a - t - b\beta - \gamma) \right) \\ &= \frac{1}{2}\beta \left(b^2\alpha^2 - 2b^2\alpha\beta + b^2\beta^2 - 2b\alpha\gamma + 2b\alpha\delta - b\alpha + 2b\beta\gamma - 2b\beta\delta + b\beta + \gamma^2 - 2\gamma\delta + \gamma\right. \\ \pi_1^{\bar{t}_2+1+} &= \frac{1}{2}\alpha \left(b^2\alpha^2 - 2b^2\alpha\beta + b^2\beta^2 - 2b\alpha\gamma + 2b\alpha\delta + b\alpha + 2b\beta\gamma - 2b\beta\delta - b\beta + \gamma^2 - 2\gamma\delta - \gamma\right. \end{aligned}$$

: If firm 1 chooses to produce in period $\bar{t}_1$firm 2 will get:

$$\left(a-\left(a-\delta-b\alpha\right)-b\left(\alpha+\beta\right)-\gamma\right)\beta=-\beta\left(\gamma-\delta+b\beta\right)$$

Thus their profit form choosing $t_2=\bar{t}_2$ is at least:

$$\begin{aligned}\underline{\pi}_2^{\bar{t}_1+} &= -\beta\left(\gamma-\delta+b\beta\right)+\frac{1}{2}\beta\left(b^2\alpha^2-2b^2\alpha\beta+b^2\beta^2-2b\alpha\gamma+2b\alpha\delta-b\alpha+2b\beta\gamma-2b\beta\delta+b\beta\right. \\ \underline{\pi}_1^{\bar{t}_2+} &= -\alpha\left(\delta-\gamma+b\alpha\right)+\frac{1}{2}\alpha\left(b^2\alpha^2-2b^2\alpha\beta+b^2\beta^2-2b\alpha\gamma+2b\alpha\delta+b\alpha+2b\beta\gamma-2b\beta\delta-b\beta\right.\end{aligned}$$

which is strictly positive. Thus they have a dominant strategy of $t_2=\bar{t}_2$. The best response of firm one is to choose $t_1=\bar{t}_1-1$, or to not produce this period.

(d) *(3 points)* Given the proof in the last step, assuming that $\min\{t_1,t_2\}\geq\min\{\bar{t}_1,\bar{t}_2\}-2$, find the unique equilibrium values of (t_1,t_2).

Solution 3 *From the last step we know that $t_1\leq\bar{t}_1-1$. Thus if firm 2 chooses $t_2=\bar{t}_2$ their profit from next period on is which is lower than before, thus they will choose $t_2=\bar{t}_2$, and thus firm one will choose $t_1=\bar{t}_1-2$.*

(e) *(4 points)* Find the unique equilibrium of the entire game, proving your conclusion.

Solution 4 *The equilibrium will be $t_2=\bar{t}_2$, $t_1=\underline{t}$, the points will be given for the argument. We proceed by iteration, assume that $\min\{t_1,t_2\}\geq t-1$ and $t_1\leq t$ for $\bar{t}_1-2>t>\underline{t}$. The profits firm 2 gets in periods $\bar{t}_1+1$ to $\bar{t}_2$ is high enough for them to be willing to take the loss in period $\bar{t}_1$, the loss in any period $t\leq\bar{t}_1$ are lower than that, thus it is always optimal to choose $t_2=\bar{t}_2$, $t_1=t-1$. Thus by iteration we are done.*

6. *(6 points)* Give a precise definition of an *empty threat*. Do empty threat's have to help the person who is making it? Why or why not?

7. *(16 points)* Consider a simultaneous move game, change it into a sequential game by having player one go first. In the subgame perfect equilibrium of the resulting game, do player one's payoffs increase, decrease, or we can't tell? Likewise for player two. Either prove your conclusion or give examples illustrating your statement.