ECON 439
Midterm: Normal Form Games

Kevin Hasker

This exam will start at 17:40 and finish at 21:20.

Answer all questions in the space provided. Points will only be given for work shown.

1. *(12 points)* Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test.

2. *(21 points total)* Consider a normal form game with a finite number of players, each of which has a finite number of strategies S_i, $\#(S_i) < \infty$.

   (a) *(4 points)* Give a mathematical definition of a mixed strategy for player i.

   $$\sigma_i \in \Sigma_i = \Delta(S_i) = \left\{ (p(s_i))_{s_i \in S_i} | \forall s_i \in S_i, p(s_i) \in [0,1], \sum_{s_i \in S_i} p(s_i) = 1 \right\}$$

   (b) *(2 points)* How does this definition describe a person's actual behavior in a game of pure conflict like Rock/Paper/Scissors?

   When I consider what I do it is much more complex than this, and doesn't resemble choosing a randomized strategy at all. I will first consider what my opponent has done in the past, seeking any patterns, then I will consider how sophisticated I think this person is, and what they might expect that I will do. I will then iterate this process until I think I have gone further than my opponent, and choose a pure strategy.

   (c) *(2 points)* What do you think of this as a model of how a person behaves in a game of pure conflict like Rock/Paper/Scissors?

   However if I was to admit it, the end point of this process is guessing that my opponent will play each action with some probability, and me simply best responding to it. Thus while it doesn't represent the process, it is an excellent reduced form description of my conclusion. This makes it a good model. **Of course for both of these questions you will have personal answers. The points will be given for reasonable argument.**

   (d) *(6 points)* Define a Nash equilibrium in (potentially) mixed strategies. (Your answer does not have to be the one I use.)

   i. *A mixed strategy equilibrium is a* $\sigma^* \in \Sigma^*$ *such that for all i there is a* $\beta_i \in \Sigma_{-i} = \times_{j \neq i} \Sigma_j$ *such that* $\sigma_i^* \in \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \beta_i)$ *and* $\beta_i = (\sigma_{-i}^*) = (\sigma_j)_{j \in I \setminus i}$.

ii. *A mixed strategy equilibrium is a* $\sigma^* \in \Sigma^*$ *such that for all* i $\sigma_i^* \in \arg\max_{\sigma_i \in \Sigma_i} u_i\left(\sigma_i, \sigma_{-i}^*\right)$.

(e) *(4 points)* What is the best method to find a strictly mixed strategy equilibrium? You might want to consider a game like matching pennies or Rock/Paper/Scissors when answering this.

If s_i *and* $\tilde{s}_i$ *both have strictly positive probability in the equilibrium, then* $u_i\left(s_i, \sigma_{-i}^*\right) = u_i\left(\tilde{s}_i, \sigma_{-i}^*\right)$.

(f) *(3 points)* If you use the method above what check do you have to do after you find your potential equilibrium? [**Hint:** You might want to think about Dinner with the Enemy or the Speeding game.]

If $\hat{s}_i$ *has zero probability in the equilibrium, and* s_i *has strictly positive probability we have to be sure that* $u_i\left(\hat{s}_i, \sigma_{-i}^*\right) \leq u_i\left(s_i, \sigma_{-i}^*\right)$. *Otherwise players will not want to take the mixed strategy they are supposed to. For example in the speeding game while the pure strategy best response for the driver is to speed when the cops aren't there, in equilibrium they never speed.*

a	b	χ_1	χ_2	q_1^*	q_2^*	Q^*	P^*	$q_{2\alpha}(q_1)$	$q_{2\beta}(q_1)$	$q_1(Eq_2)$
19	$\frac{1}{2}$	7	4	6	12	18	10	$15 - \frac{1}{2}q_1$	$12 - \frac{1}{2}q_1$	$12 - \frac{1}{2}Eq_2$
15	$\frac{1}{3}$	5	1	6	18	24	7	$21 - \frac{1}{2}q_1$	$15 - \frac{1}{2}q_1$	$15 - \frac{1}{2}Eq_2$
18	$\frac{1}{2}$	3	6	12	6	18	9	$12 - \frac{1}{2}q_1$	$15 - \frac{1}{2}q_1$	$15 - \frac{1}{2}Eq_2$
11	$\frac{1}{6}$	3	4	18	12	30	6	$21 - \frac{1}{2}q_1$	$24 - \frac{1}{2}q_1$	$24 - \frac{1}{2}Eq_2$

a	b	χ_1	χ_2	$q_{2\alpha}^*$	$q_{2\beta}^*$	q_1^*	$Q(n)$	$P(n)$	$> c_i$
19	$\frac{1}{2}$	7	4	$\tau + 11$	$\tau + 8$	$8 - 2\tau$	$24\frac{n}{n+1}$	$\frac{7n+19}{n+1}$	$\frac{7n+19}{n+1}$
15	$\frac{1}{3}$	5	1	$2\tau + 16$	$2\tau + 10$	$10 - 4\tau$	$30\frac{n}{n+1}$	$\frac{5n+15}{n+1}$	$\frac{5n+15}{n+1}$
18	$\frac{1}{2}$	3	6	$7 - \tau$	$10 - \tau$	$2\tau + 10$	$30\frac{n}{n+1}$	$\frac{3n+18}{n+1}$	$\frac{3n+18}{n+1}$
11	$\frac{1}{6}$	3	4	$13 - \tau$	$16 - \tau$	$2\tau + 16$	$48\frac{n}{n+1}$	$\frac{3n+11}{n+1}$	$\frac{3n+11}{n+1}$

3. *(36 points total)* Consider a Cournot oligopoly. Firm choose quantity, $q_i \geq 0$, and the price is based on market clearing at the total quantity produced, $Q = \sum_{i=1}^{n} q_i$: $P = a - bQ$ and all firms have the constant marginal cost, $c_i(q) = c_i q_i$; let $Q_{-i} = \sum_{j \neq i} q_j$.

(a) *(10 points total)* First consider the two firm case, where $c_1(q) = \chi_1 q_1$ and $c_2(q) = \chi_2 q_2$.

i. *(2 points)* Set up both firm's objective functions.

$$\max_{q_1} (a - b(q_1 + q_2)) q_1 - \chi_1 q_1$$
$$\max_{q_2} (a - b(q_1 + q_2)) q_2 - \chi_2 q_2$$

ii. *(4 points)* Find the first order conditions for both firms.

$$\begin{aligned}(a-b(q_1+q_2))-bq_1-\chi_1 &= 0\\(a-b(q_1+q_2))-bq_2-\chi_2 &= 0\end{aligned}$$

iii. *(6 points)* Find the quantity each firm will produce in equilibrium, the total quantity, and the market price.

$$\begin{aligned}q_1 &= \frac{1}{2b}(a-\chi_1)-\frac{1}{2}q_2\\q_2 &= \frac{1}{2b}(a-\chi_2)-\frac{1}{2}q_1\end{aligned}$$

$$\begin{aligned}q_1 &= \frac{1}{2b}(a-\chi_1)-\frac{1}{2}\left(\frac{1}{2b}(a-\chi_2)-\frac{1}{2}q_1\right)\\q_1 &= \frac{1}{3b}(a-2\chi_1+\chi_2)\\q_2 &= \frac{1}{2b}(a-\chi_2)-\frac{1}{2}\left(\frac{1}{3b}(a-2\chi_1+\chi_2)\right)\\q_2 &= \frac{1}{3b}(a+\chi_1-2\chi_2)\end{aligned}$$

$$\begin{aligned}Q &= \frac{1}{3b}(a-2\chi_1+\chi_2)+\frac{1}{3b}(a+\chi_1-2\chi_2)\\&= \frac{1}{3b}(2a-\chi_1-\chi_2)\\P &= a-b\left(\frac{1}{3b}(2a-\chi_1-\chi_2)\right)\\&= \frac{1}{3}a+\frac{1}{3}\chi_1+\frac{1}{3}\chi_2\end{aligned}$$

(b) *(10 points total)* Now assume that firm 2's costs are

$$c_2(q)=\begin{cases}\chi_2 q_2 & \text{with probability } \tau\\ \chi_1 q_1 & \text{with probability } 1-\tau\end{cases}$$

where $\tau\in[0,1]$.

i. *(4 points)* Find the best response of both types of firm 2 and firm 1. [**Hint:** There is probably a reason I am not asking you to set up the objective function again.]

Let $q_{2\alpha}$ be the quantity firm 2 will produce with marginal cost χ_2 and $q_{2\beta}$ be the quantity they will produce when they have marginal costs χ_1. Then from above I can see that:

$$\begin{aligned}q_{2\beta} &= \frac{1}{2b}(a-\chi_1)-\frac{1}{2}q_1\\q_{2\alpha} &= \frac{1}{2b}(a-\chi_2)-\frac{1}{2}q_1\end{aligned}$$

and since q_2 *was already a random variable I can see that*

$$q_1 = \frac{1}{2b}\left(a - \chi_1\right) - \frac{1}{2}Eq_2$$

where

$$Eq_2 = \tau q_{2\alpha} + \left(1 - \tau\right) q_{2\beta}$$

ii. *(6 points)* Find the quantity each firm will produce in equilibrium, the total quantity, and the market price. All answers should be a function of τ.

$$\begin{aligned}
Eq_2 &= \tau q_{2\alpha} + \left(1 - \tau\right) q_{2\beta} \\
&= \tau\left(\frac{1}{2b}\left(a - \chi_2\right) - \frac{1}{2}q_1\right) + \left(1 - \tau\right)\left(\frac{1}{2b}\left(a - \chi_1\right) - \frac{1}{2}q_1\right) \\
&= \frac{1}{2b}\left(a - \chi_1 + \tau\chi_1 - \tau\chi_2\right) - \frac{1}{2}q_1
\end{aligned}$$

$$\begin{aligned}
q_1 &= \frac{1}{2b}\left(a - \chi_1\right) - \frac{1}{2}Eq_2 \\
&= \frac{1}{2b}\left(a - \chi_1\right) - \frac{1}{2}\left(\frac{1}{2b}\left(a - \chi_1 + \tau\chi_1 - \tau\chi_2\right) - \frac{1}{2}q_1\right) \\
q_1 &= \frac{1}{4b}\left(a - \chi_1 - \tau\chi_1 + \tau\chi_2\right) + \frac{1}{4}q_1 \\
q_1 &= \frac{1}{3b}\left(a - \chi_1 - \tau\chi_1 + \tau\chi_2\right)
\end{aligned}$$

$$\begin{aligned}
q_{2\beta} &= \frac{1}{2b}\left(a - \chi_1\right) - \frac{1}{2}\left(\frac{1}{3b}\left(a - \chi_1 - \tau\chi_1 + \tau\chi_2\right)\right) \\
&= \frac{1}{6b}\left(2a - 2\chi_1 + \tau\chi_1 - \tau\chi_2\right) \\
q_{2\alpha} &= \frac{1}{2b}\left(a - \chi_2\right) - \frac{1}{2}\left(\frac{1}{3b}\left(a - \chi_1 - \tau\chi_1 + \tau\chi_2\right)\right) \\
&= \frac{1}{6b}\left(2a + \chi_1 - 3\chi_2 + \tau\chi_1 - \tau\chi_2\right)
\end{aligned}$$

$$\begin{aligned}
Q_\alpha &= \frac{1}{3b}\left(a - \chi_1 - \tau\chi_1 + \tau\chi_2\right) + \frac{1}{6b}\left(2a - 2\chi_1 + \tau\chi_1 - \tau\chi_2\right) \\
&= \frac{1}{6b}\left(4a - 4\chi_1 - \tau\chi_1 + \tau\chi_2\right) \\
P_\alpha &= a - b\left(\frac{1}{6b}\left(4a - 4\chi_1 - \tau\chi_1 + \tau\chi_2\right)\right) \\
&= \frac{1}{3}a + \frac{2}{3}\chi_1 + \frac{1}{6}\tau\chi_1 - \frac{1}{6}\tau\chi_2
\end{aligned}$$

:

$$
\begin{aligned}
Q_\beta &= \frac{1}{3b}\left(a-\chi_1-\tau\chi_1+\tau\chi_2\right)+\frac{1}{6b}\left(2a-2\chi_1+\tau\chi_1-\tau\chi_2\right) \\
&= \frac{1}{6b}\left(4a-4\chi_1-\tau\chi_1+\tau\chi_2\right) \\
P_\beta &= a-b\left(\frac{1}{6b}\left(4a-4\chi_1-\tau\chi_1+\tau\chi_2\right)\right) \\
&= \frac{1}{3}a+\frac{2}{3}\chi_1+\frac{1}{6}\tau\chi_1-\frac{1}{6}\tau\chi_2
\end{aligned}
$$

(c) *(7 points total)* Now assume there are n firms, all of which have $c_i(q)=\chi_1 q_i$. Assume that each firm produces a strictly positive level of output in equilibrium ($q_i^*>0$):

i. *(1 point)* Find the first order condition of a representative firm. *I can easily replace q_1+q_2 with Q above and so it is:*

$$(a-bQ)-bq_i-\chi_1=0$$

ii. *(6 points)* Find the total output that will be produced, the market price, and the amount produced by each firm. All answers should be a function of n. [**Hint:** There is probably a reason I don't say you can use symmetry, I will allow it but you don't need it.]
Since all firms have a strictly positive output I can sum up the n first order conditions, giving:

$$
\begin{aligned}
n(a-bQ)-b\left(\sum_i q_i\right)-n\chi_1 &= 0 \\
n(a-bQ)-bQ-n\chi_1 &= 0 \\
Q &= \frac{n}{n+1}\frac{a-\chi_1}{b}
\end{aligned}
$$

from the first order condition I see that

$$
\begin{aligned}
q_i &= \frac{a-\chi_1}{b}-Q \\
q_i &= \frac{1}{n+1}\frac{a-\chi_1}{b}
\end{aligned}
$$

$$P=\frac{a+n\chi_1}{n+1}$$

(d) *(9 points total)* Now assume there are n firms, but all you know is that the average constant marginal cost is χ_1 and that that each firm produces a strictly positive level of output in equilibrium ($q_i^*>0$):

i. *(3 points)* Find the objective function and first order condition of a representative firm. It should be a function of c_i.

$$(a-bQ)-bq_i-c_i=0$$

ii. *(4 points)* Find the total output that will be produced, the market price, and the amount produced by each firm. All answers should be a function of n, and the quantity of firm i will depend on firm i's marginal cost (c_i).
Now I am going to sum these first order conditions again and the result is:

$$(a - bQ) - b\left(\sum_i q_i\right) - \left(\sum_i c_i\right) = 0$$

and we know that $\left(\sum_i c_i\right) = n\left(\frac{1}{n}\sum_i c_i\right) = n\chi_1$. *So this becomes*

$$(a - bQ) - bQ - n\chi_1 = 0$$

which is the same as above, so

$$\begin{aligned} Q &= \frac{n}{n+1}\frac{a-\chi_1}{b} \\ P &= \frac{a+n\chi_1}{n+1} \end{aligned}$$

but now we have to figure out each firm's output individually. From the first order condition:

$$\begin{aligned} q_i &= \frac{1}{b}(a - c_i) - Q \\ q_i &= \frac{1}{b}(a - c_i) - \left(\frac{n}{n+1}\frac{a-\chi_1}{b}\right) \\ &= \frac{1}{b(n+1)}(a - c_i + n(\chi_1 - c_i)) \end{aligned}$$

iii. *(2 points)* Find an upper bound on c_i such that they will produce a strictly positive amount with n firms. What happens to this bound as $n \to \infty$? What does this tell us about market heterogeneity?
We need $q_i^* > 0$ *or*

$$\begin{aligned} \frac{1}{b(n+1)}(a - c_i + n(\chi_1 - c_i)) &> 0 \\ a - c_i + n(\chi_1 - c_i) &> 0 \\ a + n\chi_1 &> c_i(n+1) \\ \frac{a+n\chi_1}{n+1} &> c_i \\ \frac{a}{n+1} + \frac{n}{n+1}\chi_1 &> c_i \end{aligned}$$

the first term will go to zero as $n \to \infty$ *and the second term will converge to* χ_1, *thus the heterogeneity must become very small* *as n becomes larger. Notice, by the way, that $\frac{a+n\chi_1}{n+1} = P(n)$, this is not a coincidence.*

4. *(29 points total)* Consider the following Hotelling model of firm location where consumers are in discrete locations. Firms choose a location $l_i \in \{1, 2, 3, 4, 5\}$ in order to maximize the number of customers they have, and customers go to the nearest firm—splitting their business if both firms are equally close. The number of customers at each location are given by the following table.

Location	1	2	3	4	5
Number of Customers	16	2	2	6	4

Location	1	2	3	4	5
Number of Customers	4	6	2	2	16

Location	1	2	3	4	5
Number of Customers	12	2	2	8	10

Location	1	2	3	4	5
Number of Customers	10	8	2	2	12

   (a) *(6 points total)* Fill out the following table with the number of customers firm 1 has if firm 2 is at the location across the top and firm 1 is at the location in the first column.

Location	1	2	3	4	5
Number of Customers	16	2	2	6	4

Location 2→	1	2	3	4	5
Location 1↓					
1	15[1]	16[1]	17	18	19
$2(\varepsilon, 1)$	14	15	18[1]	19	20
$3(\delta, 2)$	13	12	15	20[1]	23
$4(\beta, 3)$	12	11	10	15	26[1]
$5(\alpha, 4)$	11	10	7	4	15

Location	1	2	3	4	5
Number of Customers	4	6	2	2	16

Location 2→	1	2	3	4	5
Location 1↓					
$1(\alpha, 2)$	15	4	7	10	11
$2(\beta, 3)$	26[1]	15	10	11	12
$3(\delta, 4)$	23	20[1]	15	12	13
$4(\varepsilon, 5)$	20	19	18[1]	15	14
5	19	18	17	16[1]	15[1]

Location	1	2	3	4	5
Number of Customers	12	2	2	8	10

Location 2→ / Location 1↓	1	2	3	4	5
$1(\alpha, 2)$	17	12	13	14	15
$2(\beta, 3)$	22^1	17	14	15	16
$3(\delta, 4)$	21	20^1	17	16	20
4	20	19	18^1	17^1	24^1
$5(\alpha, 4)$	19	18	14	10	17

Location	1	2	3	4	5
Number of Customers	10	8	2	2	12

Location 2→ / Location 1↓	1	2	3	4	5
$1(\alpha, 2)$	17	10	14	18	19
$2(\beta, 3)$	24^1	17^1	18^1	19	20
$3(\delta, 4)$	20	16	17	20^1	21
4	16	15	14	17	22^1
$5(\alpha, 4)$	15	14	13	12	17

(b) *(5 points total)* Find the best response for firm 1 to each location of firm 2. *They are marked in the table above with a* 1 *in the upper right hand corner of each box.*

(c) *(2 points)* Find the Nash equilibrium. *Since the best responses of both firms are the same, it is the only location where the best response to that location is that location.*

(d) *(8 points)* Show that it is the unique strategy that survives iterated deletion of dominated strategies, being clear in your argument at each step.

I will do it explicitly for this variation, above I will list which location can be deleted in the order of their deletion.

Location	1	2	3	4	5
Number of Customers	12	2	2	8	10

Location 2→ / Location 1↓	1	2	3	4	5
$1(\alpha, 2)$	17	12	13	14	15
$2(\beta, 3)$	22^1	17	14	15	16
$3(\delta, 4)$	21	20^1	17	16	20
4	20	19	18^1	17^1	24^1
$5(\alpha, 4)$	19	18	14	10	17

First we notice that only firms that are never a best response might be able to be deleted. We can guess that it will be deleted by a nearby *row, so looking at the first two rows*

1	17	12	13	14	15
2	22^1	17	14	15	16

we see that the number at location 2 is always higher than location 1, so we write $(\alpha, 2)$ *by this strategy in the table:* α *for this being one of the first strategies we can delete, and* 2 *because that dominates it (and so does* 3 *and* 4, *but I won't write them.)*

The same analysis shows that 4 dominates 5.

Now, *by the symmetry of the game we recognize this is also true for the other player, thus we can delete the first and fifth column as well as the first and fifth row.*

Location 2→	2	3	4
Location 1↓			
2	17	14	15
3	20^1	17	16
4	19	18^1	17^1

In the remaining game, 2 *is never a best response and strictly dominated by* 3 *and* 4. *Using the previous logic we delete both the row and column,*

Location 2→	3	4
Location 1↓		
3	17	16
4	18^1	17^1

and now 4 *dominates* 3.

(e) *(4 points)* Prove that in a model with a finite number of locations that if firm two is at location l_2 then the best response for firm one is either $l_2 - 1$, $l_2 + 1$, or l_2. You may assume there are a strictly positive number of customers at each location.

All that has been ruled out is being located at $l_2 + k$ *or* $l_2 - k$ *for* $k > 1$, *in this case there will be a mass of customers in between the two firms, some of which will go to the other firm. On the other hand if* $l_1 \in \{l_2 - 1, l_2, l_2 + 1\}$ *there will be no customers between the two firms that must be shared.*

(f) *(4 points)* Characterize the equilibrium for any finite number of locations, proving your assertion. You may assume the equilibrium is unique and that there are a strictly positive number of customers at each location.

Let l_m *be the location where half the customers are at that location or above, and half at that location or below. Then if* $l_2 < l_m$ *locating at* $l_2 + 1$ *guarantees that this firm gets more than half the customers. Since this is more than the opponent it is the best they can do. If*

$l_2 > l_m$ *then the same is true of* $l_2 - 1$. *If* $l_2 = l_m$ $l_1 \in \{l_2 - 1, l_2 + 1\}$ *gives strictly less than half, thus* $l_1 = l_2$ *thus the best response is:*

$$BR_1(l_2) = \begin{cases} l_2 + 1 & \text{if} \quad l_2 < l_m \\ l_2 & \text{if} \quad l_2 = l_m \\ l_2 - 1 & \text{if} \quad l_2 > l_m \end{cases}$$

since the best responses of firm 2 are symmetric, the only equilibrium is $l_1 = l_2 = l_m$.