ECON 439
Final: Extensive Form Games

Kevin Hasker

This exam will start at 15:40 and finish at 17:20.

Answer all questions in the space provided. Points will only be given for work shown.

1. *(15 points)* Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test. I recognize this means I should not use calculators or other electronic devices.

   Name and Surname: _

   Student ID: _

   Signature:

   _

2. *(8 points)* It is a well known fact that almost all professors (like I do) in the Economics department grade on the curve—i.e. the average student gets the average grade (B-/C+ in ECON 439) and then grades are assigned relative to this student. Can Strangelove is a regular student in the Economics department but has started taking his classes in the irregular semesters—i.e. spring semester classes in the following fall, etceteras. Why? I want you to explain why this is a best response for him despite it meaning that it will be harder for him to graduate on time.

   Solution Since most students are regular, most of the students taking the class in the irregular semeseters have previously flunked or (at least) done poorly in the class in the regular semester. Thus Can is selecting to take the class with stupid people and will shine by comparison, or his grade will increase and his CGPA—which of course is taken as a *signal* of his ability. ;)

3. *(9 points)* Give the definition of a *signal* in economics, and give an example from your life where you are (partially) signalling.

   Solution A *signal* is an action which does not directly maximize utility, but rather is taken to reveal or hide information about some underlying characteristic of an individual.

   Of course your example is up to you, but I bet that most of you will write: "For example I only studied for this exam in order to get a good grade and signal to my future employers I'm smart. There is nothing you are teaching me that is worth knowing!!!!" Which is perfectly fine, if incorrect, because I taught you about signalling and you just pointed out you're doing it. It's OK, we all are silly sometimes.

4. *(26 points total)* Consider T period alternating offer bargaining. Let player one's share be s_1 and player 2's share be s_2 where $s_1 \in [0,1]$, $s_2 \in [0,1]$ and $s_1 + s_2 \leq 1$. In odd periods Player 1 makes an offer of (s_1, s_2) and player 2 can either accept the offer and the negotiating terminates or rejects it and move to $t+1$ if $t < T$. In even periods player 2 makes an offer of (s_1, s_2) and player one can either accept it or reject it and move on to period $t+1$ if $t < T$. If $t = T$ then rejection means that both parties get zero. We will consider both T finite and $T = \infty$, where the bargaining may never end. Their common discount factor between periods is $\delta \in (0,1)$. **Assume that the person accepts an offer they are indifferent between accepting and rejecting.**

   (a) *(4 points)* Find the subgame perfect equilibrium when $T = 1$, or player one makes an offer and player 2 can either accept or reject it. **Hint:** the equilibrium in counter-intuitive.

   Solution We can write P2's optimal strategy as $A(x)$ and $R(y)$, we can see that since $u_2(y, R(y)) = 0$ they have to choose $A(x)$ for $x \geq 0$. Player one then has the utility $\max_{x \in A(x)} 1 - x$ thus the equilibrium is $s_1^1 = 1$, $s_2^1 = 0$.

   (b) *(4 points)* Find the subgame perfect equilibrium when $T = 2$, or if player 2 rejects player 1's offer then they can make one counter-offer.

   Solution Since we just found out what happens in the last period $\left(s_1^2, s_2^2\right) = (0,1)$ we can use this to derive that $u_2(y, R(y)) = \delta$, thus the strategy is $A(x)$ if $x \in [\delta, 1]$ and the maximum is at $\left(s_1^1, s_2^1\right) = (1-\delta, \delta)$

   (c) *(4 points)* Find the subgame perfect equilibrium when $T = 3$.

   Solution We now know that if the game goest to period 2, P2's continuation utility is $1-\delta$, thus they will reject anything that gives them less that $\delta(1-\delta)$ and $\left(s_1^1, s_2^1\right) = (1-\delta(1-\delta), \delta(1-\delta))$

   (d) *(8 points)* One can show that when $T = \infty$ there is a unique subgame perfect equilibrium, where player one offers $\left(s_1^1, s_2^1\right)$ regardless of the period and player 2 offers $\left(s_1^2, s_2^2\right)$ regardless of the period. Find these steady state values.

   Solution In the second period player 2 has to offer at least δs_1^1 to player one, or it will be rejected. As usual maximization implies that

$$s_2^2 = 1 - \delta s_1^1$$

   In the first period player 1 has to offer at least δs_2^2 to player 2 or it will be rejected, and maximization implies that:

$$\begin{aligned}
s_1^1 &= 1 - \delta s_2^2 = 1 - \delta\left(1 - \delta s_1^1\right) \\
s_1^1 &= s_1^1\delta^2 - \delta + 1 \\
\left(1 - \delta^2\right) s_1^1 &= 1 - \delta \\
s_1^1 &= \frac{1-\delta}{1-\delta^2} = \frac{1-\delta}{(1-\delta)(1+\delta)} = \frac{1}{1+\delta} \\
s_2^1 &= 1 - s_1^1 = 1 - \frac{1}{1+\delta} = \frac{\delta}{\delta+1}
\end{aligned}$$

(e) *(3 points)* Compare the subgame perfect equilibrium of this game when $T = \infty$ to experiments with the ultimatum game. Can it explain the most common outcome? Is it a reasonable explanation? Can it explain how the outcome varies across cultures? **If you are unable to find this equilibrium you can base your logic on the equilibrium when $T \in \{1, 2, 3\}$.**

Solution Theoretically, it could explain the phenomena, the modal claim in most ultimatum game experiments is between .6 and .55 so this would only need:

$$\begin{aligned}
\frac{1}{1+\delta} &= \frac{6}{10} \\
\delta &= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{1+\delta} &= \frac{55}{100} \\
\delta &= \frac{9}{11} = .8182
\end{aligned}$$

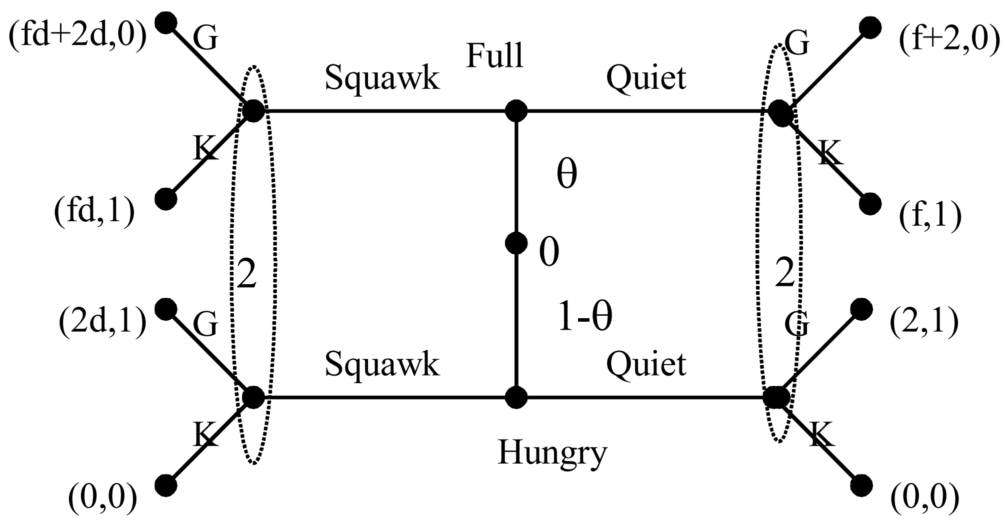
but I'm sorry to say that is emperically ridiculous. How quickly can you make a counter offer? Let's be ridiculous and say an hour? Well macroeconomists (who care about such things) say that a good annual discount factor is $\frac{1}{1+.05} = 0.952\,38$ so for an hour? For an hour that gives a discount factor of about $0.999\,99$. Or in other words the share for the first player should be $\frac{1}{1+0.999\,99} = 0.5$—or at least that's the approximation my program gives me.

Furthermore, then we need to explain the cross cultural variation. Ha ha ha ha, Ha ha ha ha, you so funny. Nope, it doesn't work. I mean, sure, we could just assume that the discounting factor differs in different populations. More generous are more patient, less generous are less patient. But please, this is a fundamental basic paramater. Don't forget it also sometimes seems to vary with stake size.

(f) *(3 points)* Explain why we must assume that a person accepts an offer if indifferent in this model, and why it is not a bad assumption. **Note that answering this question does not require answering any of the previous questions.**

Solution If the receiver rejects an offer when indifferent then $A(y) = (x, 1]$ and this set is open. Thus there is no optimal offer. This is not a big deal because if we approximate the continuum with a finite grid then $A(x) = [\lceil x \rceil, 1]$ where $\lceil x \rceil$ is the first element in the grid that is greater than or equal to x. Thus the optimal solution is $1 - \lceil x \rceil$ or $1 - x - \varepsilon$ where ε is smaller than a unit of our grid. You can see that this will approximately be x, which is why this assumption is reasonable.

5. *(30 points total)* The Sir Phillip Sidney Game: When observing colonies of birds, biologists notice that all the adult birds go out to gathering food and then when they return they feed whichever chick (baby bird) is squawking that is closely enough related to them. The following game summarizes this as a strategic interaction. First nature determines whether a chick is full (F) or hungry (H), it is full with probability $\theta \in (0, 1)$. Then the chick either squawks (S) or is quiet (Q), then the adult decides to either give the food (G) or keep it (K). The payoffs recognize that feeding the chicks costs the adult, and ignores the issue of how closely related the birds are. The cost of squawking is a discount of the benefit of being fed, their utility is discounted by $d \in (0, 1)$ if they squawk. **For your exam** $f = X$.



(a) *(4 points)* Write down all of the strategies of the adult (Player 2) and the chick (Player 1).

$$\begin{aligned} S_1 &= \{S(F), Q(F)\} \times \{S(H), Q(H)\} \\ &= \{(S(F), S(H)), (Q(F), S(H)), (S(F), Q(H)), (Q(F), Q(H))\} \end{aligned}$$

$$\begin{aligned} S_2 &= \{K(S), G(S)\} \times \{K(Q), G(Q)\} \\ &= \{(K(S), K(Q)), (G(S), K(Q)), (K(S), G(Q)), (G(S), G(Q))\} \end{aligned}$$

(b) *(8 points)* Solve the game as if the adults (Player 2) had complete information, then rewrite this using strategies you found in the last part of the question, is it a weak sequential equilibrium? If it is a weak sequential equilibrium specify the beliefs.

Solution Player 2 will choose $(G(H), K(F))$ and player 1 will always choose $(Q(F), Q(H))$. Since player 1's all do the same thing player 2 can not use the desired strategy of only feeding the hungry.

(c) *(15 points)* Fill out the following table, in the first column (s_2) should be a strategy of Player 2 (the adult). In the second column should be a Best response for player 1 (the chick) ($BR_1(s_2)$), in the next three columns should be the range for three parameters under which this is an equilibrium. These are the value for d, θ, and $\beta_1(H)$ which are the probability the adult assigns to the chick being hungry when they do what they are not expected to. If these values do not matter then you should leave them blank. Finally if this can be an equilibrium I want you to say whether it is a pooling equilibrium or a separating equilibrium. This answer should be in the last column, under *Type*?. There are more rows than necessary in this table.

s_2	$BR_1(s_2)$	d	θ	$\beta_1(H)$	*Type*?
$(K(S), K(Q))$	$(Q(F), Q(H))$	$\cdot$	$\geq \frac{1}{2}$	$\leq \frac{1}{2}$	Pooling
$(G(S), K(Q))$	$(Q(F), S(H))$	$\leq \frac{f}{f+2}$	$\cdot$	$\cdot$	Seperating
$(G(S), K(Q))$	$(S(F), S(H))$	$\geq \frac{f}{f+2}$	$\leq \frac{1}{2}$	$\leq \frac{1}{2}$	Pooling
$(K(S), G(Q))$	$(Q(F), Q(H))$	$\cdot$	$\leq \frac{1}{2}$	$\leq \frac{1}{2}$	Pooling
$(G(S), G(Q))$	$(Q(F), Q(H))$	$\cdot$	$\leq \frac{1}{2}$	$\geq \frac{1}{2}$	Pooling

Solution First let us figure out when $P2$ will choose to give as a function of the probability the chick is full. Just to be clear let's call this σ because I inadvertantly messed up the notation for β_1, it should be $\beta_1(F)$ to be consistent with θ.

$$\begin{aligned} U_2(G, \sigma) &= \sigma(0) + (1 - \sigma)(1) = 1 - \sigma \\ U_2(K, \sigma) &= \sigma(1) + (1 - \sigma)(0) = \sigma \\ U_2(G, \sigma) &\geq U_2(K, \sigma) \\ 1 - \sigma &\geq \sigma \\ \sigma &\leq \frac{1}{2} \end{aligned}$$

Thus giving is a best response when there are more full chicks than hungry ones.
Player one will remain quiet if Player 2's strategy does not depend on P1's action or if the reward for squawking (which has a cost) is less than the reward for keeping quiet. Thus $BR_1(X) = (Q(F), S(H))$ for $X \in \{(K(S), K(Q)), (K(S), G(Q)), (G(S), G(Q))\}$
In the case of $(G(S), K(Q))$ if a chick squawks it gets fed, if it does not then it does not, so the choice is:

$$U_1(G(S), S, Y) \gtreqless U_1(K((Q)), Q, Y)$$

for $Y \in \{F, H\}$. If the chick is hungry:

$$U_1(G(S), S, H) = 2d \geq 0 = U_1(K((Q)), Q, H)$$

If the chick is full:

$$U_1(G(S), S, Y) = (f+2)d \gtreqless f = U_1(K((Q)), Q, Y)$$

so S is optimal if $(f+2)d \geq f$ or $d \geq \frac{f}{f+2}$, and Q otherwise.

(d) *(3 points)* As I mentioned above, the empirical observation is that adults feed (closely related) squawking birds. Which of these equilibria does that make of interest? Recognizing that f is a parameter that might vary, discuss how this variation can explain the empirical observations more clearly, discuss how it should impact other parameters of the model.

Solution This makes the equilibria: $(G(S), K(Q))$, $\{(S(F), S(H)), (Q(F), S(H))\}$ of interest. From a biological point of view it is obvious why the birds would evolve to this equilbrium—otherwise either the parents or chicks starve to death. However if we are in the equilibrium $(S(F), S(H))$ then again the parents starve to death, so the fact that f would depend on the time since a chick last ate (a random variable) comes to the rescue. Essentially a full chick follows the cut-off strategy:

$$s_2(f) = \begin{cases} S(f) & \text{if } f \leq \frac{2d}{1-d} \\ Q(f) & \text{if } f \geq \frac{2d}{1-d} \end{cases}$$

And in this model:

$$\theta = \Pr\left(f \leq \frac{2d}{1-d}\right)$$

thus we can assume d is under evolutionary pressure to maximize the survival of the colony by optimizing the sharing of food.

6. *(12 points total)* About weak sequential equilibria.

(a) *(3 points)* Define an *assessment*.

Solution An *assessment* is a (mixed) strategy, $\sigma = [\sigma_i]_{i \in I}$ and a system of beliefs $\beta = [\beta_i]_{i \in I}$ about which history has occured.

(b) *(3 points)* Define *consistent beliefs.*

Solution β_i is *consistent* with σ if it is derived from σ using Bayes rule whenever possible.

(c) *(3 points)* Define *sequential rationality.*

Solution (σ_i, β_i) is *sequentially rational* if for all $h \in H$, $\sigma_i(h)$ is optimal given $\beta_i(h)$.

(d) *(3 points)* Define a *weak sequential equilibrium.*

Solution (σ, β) is *weak sequential equilibrium* if β are consistent with σ and σ_i is sequentially rational given β_i.

Remark Notice the elegant nature of this solution, essentially we first derive beliefs, and then given these beliefs this is nothing more than a decision problem. Of course we might find a problem with some player's strategies, which would lead us to iterate, but the beauty of this is that it is beliefs that depend on other's actions. There is no direct interaction between their strategies and mine, rather my beliefs are a function of their strategies which then are a function of mine.