ECON 439
Midterm: Normal Form Games

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This exam will start at 17:40 and finish at 19:20.

Answer all questions in the space provided. Points will only be given for work shown.

1. *(12 points)* Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test.

   Name and Surname: _
   Student ID: _
   Signature:

   _

2. *(20 points total)* Consider a second price auction where player i's values is $v_i \in [0, v^+]$. Each player submits a bid, $b_i \in [0, v^+]$, let $\bar{b}_{-i} = \max_{j \in I \setminus i} b_j$, then their utility function can be written as:

$$u_i\left(b_i, \bar{b}_{-i}\right) = \begin{cases} v_i - \bar{b}_{-i} & \text{If } b_i > \bar{b}_{-i} \\ \frac{1}{J}\left(v_i - \bar{b}_{-i}\right) & \text{If } b_i = \bar{b}_{-i} \text{ where } J = \#\left(\{j | j \in I, b_j = b_i\}\right) \\ 0 & \text{If } b_i < \bar{b}_{-i} \end{cases}$$

   (a) *(12 points total)* If $[v_i]_{i \in I}$ are common knowledge for all players:

   i. *(6 points)* Find the best response of player i. You may assume that bids are in very small units (κ, kurus) if you want but this is not necessary.

   The best response is:

$$BR_i\left(\bar{b}_{-i}\right) = \begin{cases} \left(\bar{b}_{-i}, v^+\right] & \text{if } v_i > \bar{b}_{-i} \\ [0, v^+] & \text{if } v_i = \bar{b}_{-i} \\ \left[0, \bar{b}_{-i}\right) & \text{if } v_i < \bar{b}_{-i} \end{cases}.$$

   Explaining the answer case by case, if $v_i > \bar{b}_{-i}$ *then this bidder wants to win the auction, and thus all they care about is bidding strictly more than the others, thus* $\left(\bar{b}_{-i}, v^+\right]$. *If* $v_i = \bar{b}_{-i}$ *then this bidder does not care whether they win or not, so they can bid any amount. If* $v_i < \bar{b}_{-i}$ *then if they were to win they would get* $v_i - \bar{b}_{-i} < 0$, *so that they strictly want to lose, thus* $\left[0, \bar{b}_{-i}\right)$.

   ii. *(2 points)* Find an equilibrium in weakly dominant strategies.

   This is simpler than you might think, all one has to show is that for all $\bar{b}_{-i}$, $v_i \in BR_i\left(\bar{b}_{-i}\right)$. *If* $v_i > \bar{b}_{-i}$ *then* $v_i \in \left(\bar{b}_{-i}, v^+\right]$, *if* $v_i < \bar{b}_{-i}$ *then* $v_i \in \left[0, \bar{b}_{-i}\right)$, *if* $v_i = \bar{b}_{-i}$ *then* $v_i \in [0, v^+]$. *Thus it is always a best response and weakly dominant to set* $b_i = v_i$.

iii. *(4 points)* Characterize the full set of equilibria where one player wins for sure.
We know that $b_i > \bar{b}_{-i}$ now in order to make sure that no bidder j wants to raise their bid we must have $b_i \geq \bar{v}_{-i} = \max_{j \in I \backslash i} v_j$. In order to be sure that bidder i wants to win we must be sure that $\bar{b}_{-i} \leq v_i$. Thus the set of equilibria is:
For arbitrary i, $b_i \geq \bar{v}_{-i}$ and $\bar{b}_{-i} \leq v_i$.

(b) *(8 points total)* If i knows v_i and that $[v_j]_{j \in I \backslash i}$ are i.i.d. with a distribution function $F(\cdot)$. Assume that $F(a) > 0$ for any $a \in (0, v^+)$.

i. *(2 points)* Show that the equilibrium in weakly dominant strategies is still an equilibrium.
The only change is that now $\bar{b}_{-i}$ might be a random variable, but $b_i = v_i$ is a best response for all $\bar{b}_{-i}$ thus it is an equilibrium.

ii. *(2 points)* Show that there can be an equilibrium where for arbitrary $j \in I$, $b_j = v^+$, and for all other $i \in I \backslash j$, $b_i = 0$.
Since $v^+ \geq v_i$ for all i, no i has any reason to raise their bid. Since $v_j \geq 0$ bidder j will want to win.

iii. *(4 points)* A *reservation price*, denoted r, is a minimal price at which the unit will be sold. If only one person bids more than r they buy it at the price r, if none bid more than r then no one buys the good.
Prove that if $r > 0$ then there is no equilibrium where for arbitrary $j \in I$, $b_j = v^+$, and for all other $i \in I \backslash j$, $b_i \leq r$.
All I really need to show is that this is not a best response for bidder j, but I will go farther.
First, with positive probability $v_j < r$, thus their best response must be:

$$b_j(v_j) = \begin{cases} v^+ & v_j \geq r \\ [0, r) & v_j \leq r \end{cases}$$

Given this is their best response, now with strictly positive probability some $i \neq j$ might win the auction. The only reason bidding something less than r was a best response was because there was a zero probability of winning. Thus at no point is bidding less than r a best response, indeed if $r < v_i$ then they must bid strictly more than $\max_{k \in I \backslash (i,j)} b_k \leq r$.

3. *(30 points total)* Consider the following normal form game:

Our base game is:

	α	β	δ	γ
A	$a + \frac{1}{1-q}a - b; b$	$\frac{1}{1-q}a - b; \frac{b}{p}$	$c; \frac{b}{p} - c$	$a; \frac{b}{p} - c - d$
B	$a; \frac{b}{p} - b$	$\frac{1}{1-q}a; \frac{b}{p} - 2b$	$c - 1 - d; \frac{b}{p} - 2b - 1$	$a - d; \frac{b}{p} - 2b - 1 - c$
C	$a + \frac{1}{1-q}a - b - c; a + c$	$\frac{1}{1-q}a - b - d; a - b - c$	$c - 1; a$	$a + c; a - c$
D	$\frac{1}{1-q}a - b - c; a - c$	$\frac{1}{1-q}a - b - d - a; a$	$c - 1 - d; a + d$	$a; a$

the various permutations on the exam where created to have a symmetric level of difficulty. Each pair of strategies (those in the NE, those dominated) was broken up and interwoven with the other pair.

	α	$\beta, d(\gamma)$	δ	$\gamma, id(\delta)$
$A, d(C)$	$-2;-1$	$3;3$	$-3;3$	$1;5^2$
B	$3;4^2$	$1;-3$	$4;2^1$	$1;1$
$C, id(D)$	$1;7^2$	$7;-1^1$	$0;-3$	$3;3$
D	$5;2^1$	$3;0$	$2;6^2$	$4;2^1$

a	b	c	d	p	q
3	2	4	2	$\frac{1}{3}$	$\frac{1}{4}$

	$\alpha, d(\delta)$	β	$\delta, id(\beta)$	γ
A	$3;1$	$8;6^2$	$2;4^1$	$9;4^1$
$B, d(D)$	$3;3$	$2;3$	$-2;6^2$	$6;1$
C	$0;-5$	$12;-2^1$	$-2;-3$	$3;2^2$
$D, id(A)$	$5;1^1$	$5;-3$	$1;3$	$7;5^2$

a	b	c	d	p	q
3	4	2	3	$\frac{2}{3}$	$\frac{3}{4}$

	α	$\beta, d(\gamma)$	δ	$\gamma, id(\alpha)$
A	$3;6^1$	$0;0$	$0;9^2$	$3;5$
$B, id(C)$	$-1;-7$	$6;-4^1$	$-4;6^2$	$4;1$
C	$0;12^2$	$1;6$	$1;3^1$	$5;7^1$
$D, d(B)$	$-2;1$	$1;1$	$-5;-4$	$3;2^2$

a	b	c	d	p	q
1	3	5	1	$\frac{1}{4}$	$\frac{1}{4}$

	$\alpha, d(\delta)$	β	$\delta, id(\beta)$	γ
A	$2;-6$	$4;1^2$	$0;-3$	$6;-2^1$
$B, d(D)$	$4;4$	$0;1$	$0;6^2$	$-3;4$
C	$4;-1$	$7;3^1$	$3;1^1$	$3;4^2$
$D, id(C)$	$7;1^1$	$4;7^2$	$2;4$	$1;-2$

a	b	c	d	p	q
4	3	3	2	$\frac{3}{4}$	$\frac{1}{3}$

(a) *(6 points)* Explain, in detail, how you find and what are the best responses to B and β. *I will use the game:*

	α	$\beta, d(\gamma)$	δ	$\gamma, id(\alpha)$
A	$3;6^1$	$0;0$	$0;9^2$	$3;5$
$B, id(C)$	$-1;-7$	$6;-4^1$	$-4;6^2$	$4;1$
C	$0;12^2$	$1;6$	$1;3^1$	$5;7^1$
$D, d(B)$	$-2;1$	$1;1$	$-5;-4$	$3;2^2$

The best response to B in this game is δ, because $u_2(B,\delta) = 6 > 1 = u_2(B,\gamma) > -4 = u_2(B,\beta) > -7 = u_2(B,\alpha)$. Likewise the best strategy for player 1 against β is B, because $u_1(B,\beta) > \max(u_1(A,\beta), u_1(C,\beta), u_1(D,\beta))$.

(b) *(6 points)* Find all the other pure strategy best responses for both players. You may mark them on the table above but you will loose four points if you do not explain your notation below.

My notation is I put a 1 (respectively, 2) in the upper right hand corner if it is a best response for player 1 (respectively, 2).

(c) *(6 points)* In this game there is one dominated strategy for each player. What is it, what dominates it, and why?

In each game the notation is $X, d(Y)$, which means that Y dominates X. To explain why I will explain why B dominates D in this game:

	α	$\beta, d(\gamma)$	δ	$\gamma, id(\alpha)$
A	$3; 6^1$	$0; 0$	$0; 9^2$	$3; 5$
$B, id(C)$	$-1; -7$	$6; -4^1$	$-4; 6^2$	$4; 1$
C	$0; 12^2$	$1; 6$	$1; 3^1$	$5; 7^1$
$D, d(B)$	$-2; 1$	$1; 1$	$-5; -4$	$3; 2^2$

this is because $u_1(B,\alpha) = -1 > u_1(D,\alpha) = -2$, $u_1(B,\beta) > u_1(D,\beta)$, $u_1(B,\delta) = -4 > -5 = u_1(D,\delta)$, and $u_1(B,\gamma) > u_1(D,\gamma)$, or B gets a strictly higher payoff against every strategy than D. Likewise γ always gets a strictly higher payoff than β.

(d) *(4 points)* In this game there are two strategies that can be ruled out by iterated removal of dominated strategies, which strategies are these and what dominates each one?

These are denoted $X, id(Y)$ meaning that Y dominates X after some strategies are removed (by dominance or iterated dominance) for the other player.

(e) *(8 points)* Write the remaining game in the table below and find the Nash equilibrium.

Just to illustrate the steps I will use the game:

	α	$\beta, d(\gamma)$	δ	$\gamma, id(\alpha)$
A	$3; 6^1$	$0; 0$	$0; 9^2$	$3; 5$
$B, id(C)$	$-1; -7$	$6; -4^1$	$-4; 6^2$	$4; 1$
C	$0; 12^2$	$1; 6$	$1; 3^1$	$5; 7^1$
$D, d(B)$	$-2; 1$	$1; 1$	$-5; -4$	$3; 2^2$

which reduced to:

	α	δ
A	$3; 6^1$	$0; 9^2$
C	$0; 12^2$	$1; 3^1$

Notice that all I really needed to know to figure this out was the statement that there was one Nash equilibrium and the fact that this *is the unique best response cycle in this game. Given the excessively useful hint that the game is supposed to be two by two after iterated deletion of dominated strategies, this must be the strategies of interest. Let* $p = \Pr_1(A)$ *and* $q = \Pr_2(\alpha)$ *then:*

$$\begin{aligned} u_1(A, q) &= 3q + 0(1-q) \\ u_1(C, q) &= 0q + 1(1-q) \end{aligned}$$

in equilibrium:

$$3q + 0(1-q) = 0q + 1(1-q)$$

or $q = \frac{1}{4}$, *which is written below the game above. We also know that*

$$\begin{aligned} u_2(\alpha, p) &= 6p + 12(1-p) \\ u_2(\delta, p) &= 9p + 3(1-p) \\ 6p + 12(1-p) &= 9p + 3(1-p) \end{aligned}$$

or $p = \frac{3}{4}$, *below the table it says* $\frac{1}{4}$, *but this just means that in the original orientation of the equilibrium I was solving for* $1 - p$.

a	b	$q_i(p_i, p_j)$	χ	τ	$p_1(p_2)$	p_a^*	$p_2(p_1, \tau)$	p_{1b}^*	p_{2b}^*	p_{1c}^*	$p_{2c}^*(\chi$
16	2	$p_j - 2p_i + 16$	2	10	$\frac{1}{4}p_2 + 5$	$\frac{20}{3}$	$\frac{1}{4}p_1 + 9$	$\frac{116}{15}$	$\frac{164}{15}$	$\frac{116}{15} - \frac{16}{15}\mu$	$\frac{104}{15} -$
20	2	$p_j - 2p_i + 20$	2	10	$\frac{1}{4}p_2 + 6$	8	$\frac{1}{4}p_1 + 10$	$\frac{136}{15}$	$\frac{184}{15}$	$\frac{136}{15} - \frac{16}{15}\mu$	$\frac{124}{15} -$
18	2	$p_j - 2p_i + 18$	12	4	$\frac{1}{4}p_2 + \frac{21}{2}$	14	$\frac{1}{4}p_1 + \frac{13}{2}$	$\frac{194}{15}$	$\frac{146}{15}$	$\frac{16}{15}\mu + \frac{194}{15}$	$\frac{4}{15}\mu +$
13	2	$p_j - 2p_i + 13$	9	1	$\frac{1}{4}p_2 + \frac{31}{4}$	$\frac{31}{3}$	$\frac{1}{4}p_1 + \frac{15}{4}$	$\frac{139}{15}$	$\frac{91}{15}$	$\frac{16}{15}\mu + \frac{139}{15}$	$\frac{4}{15}\mu +$
25	2	$p_j - 2p_i + 25$	16	1	$\frac{1}{4}p_2 + \frac{57}{4}$	19	$\frac{1}{4}p_1 + \frac{27}{4}$	17	11	$2\mu + 17$	$\frac{1}{2}\mu +$

Due to an error in my calculations, the answers are not integers in this part. My apologies, but since the calculations were so complicated I often just checked to see that you had correct technique, so in fact it did not cost you.

Just for the fun of it, I included a third set of coefficients ($a = 25$, $\chi = 16$, $\tau = 1$). *For these coefficients the answers are integers.*

4. *(38 points total)* Consider a differentiated Bertrand duopoly. Firm $i \in \{1, 2\}$ chooses a price, $p_i \in [0, \infty)$ and firm i's demand is:

$$q_i = a - 2p_i + p_j$$

where $j = \{1, 2\} \backslash i$, or the price of the other firm. Firm one has the cost $c(q) = \chi q_1$. **HINTS: In this question what I do *not* ask contains information. Also, there is a simple way that it seems you can reformulate this problem. If you do this you will get zero points.**

Just to explain the hint, one could solve for price as a function of quanitity, which would be:

$$p_1 = a - \frac{2}{3}q_1 - \frac{1}{3}q_2$$

$$\begin{aligned}
\pi(q_i, q_j) &= \left(18 - \frac{2}{3}q_i - \frac{1}{3}q_j\right) q_i - 12q_i \\
\frac{\partial \pi_i}{\partial q_i} &= 0 = \left(18 - \frac{2}{3}q_i - \frac{1}{3}q_j\right) - \frac{2}{3}q_i - 12
\end{aligned}$$

$$\begin{aligned}
q_i &= \frac{9}{2} - \frac{1}{4}q_j \\
q &= \frac{9}{2} - \frac{1}{4}q \\
q &= \frac{18}{5}
\end{aligned}$$

$$p_i = 18 - \frac{2}{3}\left(\frac{18}{5}\right) - \frac{1}{3}\left(\frac{18}{5}\right) = 14.4$$

not the 14 it is in a differentiated Bertrand game.

and it would seem that you could then solve it as a Cournot problem and everything would work out fine. But it won't, and if you did this despite me telling you not to you will get zero on the entire question.[1]

(a) (11 points total) *Symmetric: Assume that firm 2's costs are* $c(q) = \chi q_2$.

i. (2 points) *Set up the objective function of a representative firm.*

$$\max_{p_i} (p_i - \chi)(a - 2p_i + p_j)$$

ii. (3 points) *Find it's first order condition.*

$$(a - 2p_i + p_j) - 2(p_i - \chi) = 0$$

[1] Just to prove the point, let me solve for the symmetric equilibrium when $a = 18$, $\chi = 12$.

$$\begin{aligned}
\pi(q_i, q_j) &= \left(18 - \frac{2}{3}q_i - \frac{1}{3}q_j\right) q_i - 12q_i \\
\frac{\partial \pi_i}{\partial q_i} &= 0 = \left(18 - \frac{2}{3}q_i - \frac{1}{3}q_j\right) - \frac{2}{3}q_i - 12
\end{aligned}$$

$$\begin{aligned}
q_i &= \frac{9}{2} - \frac{1}{4}q_j \\
q &= \frac{9}{2} - \frac{1}{4}q \\
q &= \frac{18}{5}
\end{aligned}$$

$$p_i = 18 - \frac{2}{3}\left(\frac{18}{5}\right) - \frac{1}{3}\left(\frac{18}{5}\right) = 14.4$$

not the 14 it is in a differentiated Bertrand game.

iii. (3 points) *Find the best response of this representative firm.*

$$p_i = \frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}p_j$$

iv. (3 points) *Find the Nash equilibrium prices, you may assume the equilibrium is symmetric.*

$$\begin{aligned} p_i &= p_j = p \\ p &= \frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}p \\ p &= \frac{1}{3}a + \frac{2}{3}\chi \end{aligned}$$

(b) (12 points total) *Asymmetric: Assume that firm 2's costs are $c(q) = \tau q_2$.*

i. (2 points) *Set up the objective function of firm two.*

$$\max_{p_2} (p_2 - \tau)(a - (2)p_i + p_1)$$

ii. (3 points) *Find its first order condition.*

$$(a - 2p_2 + p_1) - 2(p_2 - \tau) = 0$$

iii. (3 points) *Find its best response.*

$$p_2 = \frac{1}{4}a + \frac{1}{2}\tau + \frac{1}{4}p_1$$

iv. (4 points) *Find the Nash equilibrium prices, you may assume the equilibrium is symmetric.*

$$\begin{aligned} p_2 &= \frac{1}{4}a + \frac{1}{2}\tau + \frac{1}{4}p_1 \\ p_1 &= \frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}p_2 \end{aligned}$$

$$\begin{aligned} p_2 &= \frac{1}{4}a + \frac{1}{2}\tau + \frac{1}{4}\left(\frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}p_2\right) \\ p_2 &= \frac{1}{3}a + \frac{8}{15}\tau + \frac{2}{15}\chi \end{aligned}$$

$$\begin{aligned} p_1 &= \frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}\left(\frac{1}{3}a + \frac{8}{15}\tau + \frac{2}{15}\chi\right) \\ p_1 &= \frac{1}{3}a + \frac{2}{15}\tau + \frac{8}{15}\chi \end{aligned}$$

(c) (15 points total) *Bayesian: Assume that firm one only knows that firm 2's costs are:*

$$c(q) = \begin{cases} \chi q_2 & \textit{with probability } \mu \\ \tau q_2 & \textit{with probability } 1-\mu \end{cases},$$

and firm 2 knows their own costs.

i. (2 points) *Carefully set up the objective function for firm one. Explain why you do not need to find it's best response again.*
The only difference from before is that p_2 is now (explicitly) a random variable. Thus the objective function is:

$$\max_{p_1} E_{p_2} (p_1 - \chi)(a - (2) p_1 + p_2)$$

since the function is linear in p_2 we can take the expectation through and get:

$$\max_{p_1} (p_1 - \chi)(a - (2) p_1 + E[p_2])$$

but this is just the same as in the case of certainty, other than that p_2 is replaced with $E[p_2]$, thus the best response must be:

$$p_1 = \frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}E[p_2]$$

ii. (2 points) *Find the expected best response of firm two, it should be a function of μ.*

$$\begin{aligned} E[p_2] &= \mu p_2(\chi) + (1-\mu) p_2(\tau) \\ &= \mu\left(\frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}p_1\right) + (1-\mu)\left(\frac{1}{4}a + \frac{1}{2}\tau + \frac{1}{4}p_1\right) \\ &= \frac{1}{4}a + \frac{1}{2}\tau + \frac{1}{4}p_1 - \frac{1}{2}\tau\mu + \frac{1}{2}\mu\chi \end{aligned}$$

iii. (3 points) *Find the equilibrium price of firm one, it should be a function of μ.*

$$\begin{aligned} p_1 &= \frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}E[p_2] \\ E[p_2] &= \frac{1}{4}a + \frac{1}{2}\tau + \frac{1}{4}p_1 - \frac{1}{2}\tau\mu + \frac{1}{2}\mu\chi \end{aligned}$$

$$p_1 = \frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}\left(\frac{1}{4}a + \frac{1}{2}\tau + \frac{1}{4}p_1 - \frac{1}{2}\tau\mu + \frac{1}{2}\mu\chi\right)$$

$$p_1 = \frac{1}{3}a + \frac{2}{15}\tau + \frac{8}{15}\chi - \frac{2}{15}\tau\mu + \frac{2}{15}\mu\chi$$

iv. (2 points) *Find the equilibrium prices for firm two, they should be functions of μ.*

$$\begin{aligned} p_2(\chi) &= \frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}p_1 \\ p_2(\tau) &= \frac{1}{4}a + \frac{1}{2}\tau + \frac{1}{4}p_1 \\ p_1 &= \frac{1}{3}a + \frac{2}{15}\tau + \frac{8}{15}\chi - \frac{2}{15}\tau\mu + \frac{2}{15}\mu\chi \end{aligned}$$

$$\begin{aligned} p_2(\chi) &= \frac{1}{4}a + \frac{1}{2}\chi + \frac{1}{4}\left(\frac{1}{3}a + \frac{2}{15}\tau + \frac{8}{15}\chi - \frac{2}{15}\tau\mu + \frac{2}{15}\mu\chi\right) \\ p_2(\tau) &= \frac{1}{4}a + \frac{1}{2}\tau + \frac{1}{4}\left(\frac{1}{3}a + \frac{2}{15}\tau + \frac{8}{15}\chi - \frac{2}{15}\tau\mu + \frac{2}{15}\mu\chi\right) \end{aligned}$$

$$\begin{aligned} p_2(\chi) &= \frac{1}{3}a + \frac{1}{30}\tau + \frac{19}{30}\chi - \frac{1}{30}\tau\mu + \frac{1}{30}\mu\chi \\ p_2(\tau) &= \frac{1}{3}a + \frac{8}{15}\tau + \frac{2}{15}\chi - \frac{1}{30}\tau\mu + \frac{1}{30}\mu\chi \end{aligned}$$

$: :$ while the abstract answers are a mess, if you go through the steps you will find that the equilibria above all fit these insane formula. Yes, it was hard finding four relatively equivalent problems. In fact, on the day of the exam I realized my original problems all resulted in "equilibrium" prices that were below marginal cost... so I had to find new coefficients for the problems.