ECON 439
Midterm: Normal Form Games

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This exam will start at or after 15:35 and must be turned into Moodle by 17:25

Points will only be given for work shown.

1. *(20 points)* Please write the following statement on the first page of your answers and below it write your full name, student ID, and sign it.

   I promise that I have neither given nor received aid from another person while taking this final exam. The following answers are in my own words and based on my own work using class notes, books, and online resources but no human assistance.

2. *(6 points)* In your own words prove that if in a Nash equilibrium the probability of $s_i \in S_i$ and $\tilde{s}_i \in S_i$ are both strictly positive and the other players strategy is $\sigma_{-i}^* \in \times_{j \neq i} \Delta(S_j)$ then $u_i(s_i, \sigma_{-i}^*) = u_i(\tilde{s}_i, \sigma_{-i}^*)$.

   Proof. Let σ_i^* be player i's mixed strategy in this equilibrium. Assume $u_i(s_i, \sigma_{-i}^*) > u_i(\tilde{s}_i, \sigma_{-i}^*)$ and consider the strategy where for $s_i' \notin \{s_i, \tilde{s}_i\}$ $\tilde{\sigma}_i(s_i') = \sigma_i^*(s_i')$ and $\tilde{\sigma}_i(s_i) = \sigma_i^*(s_i) + \sigma_i^*(\tilde{s}_i)$. Clearly this will yield a higher expected utility, contradicting σ_i^* was a rational strategy. ■

3. *(24 points total)*

	α	β	γ
A	$1;3$	$10;4^2$	$10;0^1$
B	$4;1^1$	$18;4^2$	$9;0$
C	$2;4$	$28;1^1$	$1;5^2$

   (a) *(4 points)* Carefully explain (as if to a high school student) what the best response of player 2 is to A. For all of the other pure strategies either write the best response below or mark them on the table above—but if you use the table you will loose a point if you do not explain your notation below.

   Solution 1 *The best response to A is β because $u_2(A, \beta) = 4 > 3 = u_2(A, \alpha) > u_2(A, \gamma) = 0$. For all other best responses I mark a 1 (2 respectively) if it is a best response of player 1 (2 respectively) in the table above.*

   (b) *(4 points)* Write down the cycle in pure strategy best responses below, and in a table draw the game with these strategies deleted. The table should look like:

   Solution 2 *The cycle is $(A, \beta) \to (C, \beta) \to (C, \gamma) \to (A, \gamma) \to (A, \beta)$ and the game is:*

	β	γ
A	$10;4^2$	$10;0^1$
C	$28;1^1$	$1;5^2$

(c) *(4 points)* Find a *candidate* for a Nash equilibrium where only strategies in the cycle over pure strategy best responses are given a positive probability of being used.

Solution 3 *Let* $\Pr(\beta) = q$ *then*

$$\begin{aligned} U_1(A,q) &= 10q + 10(1-q) = 10 \\ U_1(C,q) &= 28q + (1-q) = 27q + 1 \\ 27q + 1 &= 10 \\ q &= \frac{1}{3} \end{aligned}$$

and note that $U_1(p,q) = 10$. *Let* $\Pr(A) = p$ *then:*

$$\begin{aligned} U_2(p,\beta) &= 4p + (1-p) = 3p + 1 \\ U_2(p,\gamma) &= 0p + 5(1-p) = 5 - 5p \end{aligned}$$

$$\begin{aligned} 3p + 1 &= 5 - 5p \\ p &= \frac{1}{2} \end{aligned}$$

and notice that $U_2(p,q) = 3 * \frac{1}{2} + 1 = \frac{5}{2}$

(d) *(2 points)* Show that this candidate Nash equilibrium is not a Nash equilibrium of this game.

$$\begin{aligned} U_1(B,q) &= 18 * \frac{1}{3} + 9 * \left(1 - \frac{1}{3}\right) = 12 > 10 \\ U_2(p,\alpha) &= \frac{1}{2} * 3 + \frac{1}{2} * 4 = \frac{7}{2} > \frac{5}{2} \end{aligned}$$

(e) *(10 points)* Find the unique Nash equilibrium of this game.

Solution 4 *Let* $q_\alpha = \Pr(\alpha)$ *and* $q_\beta = \Pr(\beta)$ *then:*

$$\begin{aligned} U_1(A,q_\alpha,q_\beta) &= q_\alpha(1) + q_\beta(10) + (1-q_\alpha-q_\beta)(10) = 10 - 9q_\alpha \\ U_1(B,q_\alpha,q_\beta) &= q_\alpha(4) + q_\beta(18) + (1-q_\alpha-q_\beta)(9) = 9q_\beta - 5q_\alpha + 9 \\ U_1(C,q_\alpha,q_\beta) &= q_\alpha(2) + q_\beta(28) + (1-q_\alpha-q_\beta)(1) = q_\alpha + 27q_\beta + 1 \end{aligned}$$

Further allow $p_A = \Pr(A)$ *and* $p_B = \Pr(B)$ *then:*

$$\begin{aligned} U_2(p_A,p_B,\alpha) &= p_A(3) + p_B(1) + (1-p_A-p_B)(4) = 4 - 3p_B - p_A \\ U_2(p_A,p_B,\beta) &= p_A(4) + p_B(4) + (1-p_A-p_B)(1) = 3p_A + 3p_B + 1 \\ U_2(p_A,p_B,\gamma) &= p_A(0) + p_B(0) + (1-p_A-p_B)(5) = 5 - 5p_B - 5p_A \end{aligned}$$

Given I have worked on and been frustrated in finding the answer I will solve for the best response regions:

$$\begin{aligned} U_1(A, q_\alpha, q_\beta) &\geq U_1(B, q_\alpha, q_\beta) \\ 10 - 9q_\alpha &\geq 9q_\beta - 5q_\alpha + 9 \\ 10 - (9q_\beta + 9) &\geq -5q_\alpha - (-9q_\alpha) \\ 1 - 9q_\beta &\geq 4q_\alpha \\ \frac{1}{4} - \frac{9}{4}q_\beta &\geq q_\alpha \end{aligned}$$

$$\begin{aligned} U_1(A, q_\alpha, q_\beta) &\geq U_1(C, q_\alpha, q_\beta) \\ 10 - 9q_\alpha &\geq q_\alpha + 27q_\beta + 1 \\ 10 - (27q_\beta + 1) &\geq q_\alpha - (-9q_\alpha) \\ 9 - 27q_\beta &\geq 10q_\alpha \\ \frac{9}{10} - \frac{27}{10}q_\beta &\geq q_\alpha \end{aligned}$$

since $q_\alpha \geq 0$ *we know that* $q_\beta \leq \frac{1}{9}$ *(from the first equation) we note that:*

$$\begin{aligned} \frac{9}{10} - \frac{27}{10}q_\beta &\geq \frac{1}{4} - \frac{9}{4}q_\beta \\ \frac{13}{9} &\geq q_\beta \end{aligned}$$

So the region is characterized by $\frac{1}{4} - \frac{9}{4}q_\beta \geq q_\alpha$ *and* $q_\beta \leq \frac{1}{9}$*. There is a second important thing to recognize here, which is that* A *is better than* B *then it is better than* C*. Above we tried to find an equilibrium where* $p_B = 0$ *and we failed. Thus we conclude that* $p_B > 0$ *or* B *must be a best response.*

$$\begin{aligned} U_1(B, q_\alpha, q_\beta) &\geq U_1(A, q_\alpha, q_\beta) \\ q_\alpha &\geq \frac{1}{4} - \frac{9}{4}q_\beta \end{aligned}$$

$$\begin{aligned} U_1(B, q_\alpha, q_\beta) &\geq U_1(C, q_\alpha, q_\beta) \\ 9q_\beta - 5q_\alpha + 9 &\geq q_\alpha + 27q_\beta + 1 \\ 9q_\beta + 9 - (27q_\beta + 1) &\geq q_\alpha - (-5q_\alpha) \\ 8 - 18q_\beta &\geq 6q_\alpha \\ \frac{4}{3} - 3q_\beta &\geq q_\alpha \end{aligned}$$

Thus we must have $q_\beta \leq \frac{4}{9}$ *and* $\frac{4}{3} - 3q_\beta \geq q_\alpha \geq \frac{1}{4} - \frac{9}{4}q_\beta$

Finally

$$\begin{aligned} U_1(C, q_\alpha, q_\beta) &\geq U_1(A, q_\alpha, q_\beta) \\ q_\alpha &\geq \frac{9}{10} - \frac{27}{10} q_\beta \end{aligned}$$

$$\begin{aligned} U_1(C, q_\alpha, q_\beta) &\geq U_1(B, q_\alpha, q_\beta) \\ q_\alpha &\geq \frac{4}{3} - 3q_\beta \end{aligned}$$

This means we must have $q_\alpha \geq \frac{9}{10} - \frac{27}{10} q_\beta$. *Like before this means that if* C *is better than* A *is must also be better than* B.
If we are going to have all three be best responses then

$$\begin{aligned} q_\alpha &= \frac{9}{10} - \frac{27}{10} q_\beta \\ q_\alpha &= \frac{4}{3} - 3q_\beta \end{aligned}$$

$$\frac{9}{10} - \frac{27}{10} q_\beta = \frac{4}{3} - 3q_\beta, q_\beta = \frac{13}{9} > 1$$

Thus we can not have all three be best responses.
Above we tried to find an equilibrium where $p_B = 0$ *and failed, thus* $p_B > 0$ *and we must have* $q_\beta \leq \frac{4}{9}$ *and* $\frac{4}{3} - 3q_\beta \geq q_\alpha \geq \frac{1}{4} - \frac{9}{4} q_\beta$. *Thus we either have* $\frac{4}{3} - 3q_\beta = q_\alpha$ $(p_A = 0)$ *or we have* $q_\alpha = \frac{1}{4} - \frac{9}{4} q_\beta$ $(p_A + p_B = 1)$.
Analyzing person 2 *if* $p_A = 0$

$$\begin{aligned} U_2(0, p_B, \alpha) &= (0)(3) + p_B(1) + (1 - (0) - p_B)(4) = 4 - 3p_B - (0) = 4 - 3p_B \\ U_2(0, p_B, \beta) &= (0)(4) + p_B(4) + (1 - (0) - p_B)(1) = 3(0) + 3p_B + 1 = 3p_B + 1 \\ U_2(0, p_B, \gamma) &= (0)(0) + p_B(0) + (1 - (0) - p_B)(5) = 5 - 5p_B - 5(0) = 5 - 5p_B \end{aligned}$$

$$\begin{aligned} U_2(0, p_B, \alpha) &\geq U_2(0, p_B, \beta) \\ 4 - 3p_B &\geq 3p_B + 1 \\ \frac{1}{2} &\geq p_B \\ U_2(0, p_B, \alpha) &\geq U_2(0, p_B, \gamma) \\ 4 - 3p_B &\geq 5 - 5p_B \\ \frac{1}{2} &\geq p_B \end{aligned}$$

and thus if $p_B = \frac{1}{2}$ *then we have* $U_2(0, p_B, \alpha) = U_2(0, p_B, \beta) = U_2(0, p_B, \gamma)$ *and we are completely unconstrained for player 2's mixed strategy.*

The other case is $p_A = 1 - p_B$ *and in this case:*

$$\begin{aligned} U_2(1-p_B, p_B, \alpha) &= (1-p_B)(3) + p_B(1) + (1-(1-p_B)-p_B)(4) = 4 - 3p_B - (1-p_B) = \\ U_2(1-p_B, p_B, \beta) &= (1-p_B)(4) + p_B(4) + (1-(1-p_B)-p_B)(1) = 3(1-p_B) + 3p_B + 1 = \\ U_2(1-p_B, p_B, \gamma) &= (1-p_B)(0) + p_B(0) + (1-(1-p_B)-p_B)(5) = 5 - 5p_B - 5(1-p_B) = \end{aligned}$$

Thus this can not be the case. Thus we conclude that the mixed strategy NE for player 1 is $(p_A, p_B) = \left(0, \frac{1}{2}\right)$ *and for player 2 it is any* (q_α, q_β) *such that:* $\frac{4}{3} - 3q_\beta = q_\alpha \geq \frac{1}{4} - \frac{9}{4}q_\beta$ *and* $q_\alpha + q_\beta \leq 1$ *this requires that* $q_\beta \geq \frac{1}{6}$ *which means the lower constraint is non-binding thus the most elegant characterization is*

$$(p_A, p_B) = \left(0, \frac{1}{2}\right), (q_\alpha, q_\beta) = \left\{ \left(\frac{4}{3} - 3q_\beta, q_\beta\right) | q_\beta \geq \frac{1}{6} \right\}.$$

Remark 5 *So, what happened here? Clearly I goofed up. Since this is an online exam I wanted an equilibrium with three or more strategies played, but in order to also give you a simpler task I wanted a candidate with only two strategies in the support. Combining the two led to a terrible snafu. I got the candidate fine, but also generated a game with a continuum of equilibria. Heh. Well... you know... I told you during the exam to avoid that question and suggested heavy partial credit—which I did give.*

Remark 6 *Do you know how happy I am to know that the existence of a Nash equilibrium is guaranteed? I didn't need to sweat it, I knew it existed. I just didn't realize there was a continuum. I constructed the game assuming* $(q_\alpha, q_\beta) = \left(\frac{1}{3}, \frac{1}{3}\right)$ *which is* a *Nash equilibrium.*

4. *(28 points in total)* Consider the following Hotelling model. Two firms choose $l_i \in \{1, 2, 3, 4, 5\}$ to maximize the number of customers they have. Customers at location l either go to the closer firm or split their business if both firms are equally close to them. Let $D_i(l_i, l_j)$ be the number of customers firm i has if they are located at l_i and the other firm is located at l_j. The distribution of customers is:

Locations :	1	2	3	4	5
# Customers :	4	8	2	2	10

   (a) *(5 points)* Copy the following table into your answers and fill it out with the number of customers firm 1 will have if firm 2 is in location l_2. (The column is the location of firm 2, the row the location of firm 1.)

If $l_2 =$	1	2	3	4	5
$D_1(1, l_2) =$	13	4	8	12	13
$D_1(2, l_2) =$	22	13	12	13	14
$D_1(3, l_2) =$	18	14	13	14	15
$D_1(4, l_2) =$	14	13	12	13	16
$D_1(5, l_2) =$	13	12	11	10	13

(b) *(6 points)* From the table what are the best responses of firm 1 to firm 2 and the Nash equilibrium? Why do you not have to do a new table for firm 2?

Solution 7

If $l_2 =$	*1*	*2*	*3*	*4*	*5*
$BR_1(l_2) =$	2	3	3	3	4

and the Nash equilibrium is $l_1 = l_2 = 3$. *We do not have to do a new table for firm 2 because they have a symmetric objective function.*

(c) *(8 points)* Solve this game by iterated deletion of dominated strategies.

Solution 8

If $l_2 =$	*1*	*2*	*3*	*4*	*5*
$D_1(1, l_2) =$	13	4	8	12	13
$D_1(2, l_2) =$	22	13	12	13	14
$D_1(3, l_2) =$	18	14	13	14	15
$D_1(4, l_2) =$	14	13	12	13	16
$D_1(5, l_2) =$	13	12	11	10	13

We can see $D_1(2, l_2) > D_1(1, l_2)$ *and also* $D_1(4, l_2) > D_1(5, l_2)$ *this means that firm 2 will never locate at either of these locations and the remaining problem is:*

If $l_2 =$	*2*	*3*	*4*
$D_1(2, l_2) =$	13	12	13
$D_1(3, l_2) =$	14	13	14
$D_1(4, l_2) =$	13	12	13

and in this game we can see that $D_1(3, l_2) > D_1(2, l_2)$ *and that* $D_1(3, l_2) > D_1(4, l_2)$ *thus the only location to survive iterated deletion of dominated strategies is* $l_i = 3$.

From here on consider a generic version of this model, $l_i \in \{1, 2, 3, ..., L\}$ and assume that the median location (l_m) is unique.

(d) *(6 points)* Assume that firm i and j can choose a location from the set $\{l_-, l_- + 1, l_- + 2, ..., l_+\}$. Show that if $l_+ > l_m$ then $l_+ - 1$ dominates l_+, and likewise if $l_- < l_m$ $l_- + 1$ dominates l_-.

Proof. Let C be the total number of customers, since $l_+ > l_m$ we know that $D_i(l_+ - 1, l_+) > \frac{C}{2} > D_i(l_+, l_+ - 1)$ because the firm at $l_+ - 1$ will get the customers at l_m and below. Notice that since $D_i(l_+ - 1, l_+ - 1) = \frac{C}{2}$ this also establishes that $D_i(l_+ - 1, l_+ - 1) > D_i(l_+, l_+ - 1)$. Now for every $l_j \in \{l_-, l_- + 1, l_- + 2, ..., l_+ - 2\}$ a firm at $l_+ - 1$ is closer than a firm at l_+, thus *as long as their are* *customers in every location* $D_i(l_+ - 1, l_j) > D_i(l_+, l_j)$. The proof for $l_- < l_m$ is constructed with a symmetric argument, thus the proof is done.

I must admit that I didn't realize when writing this that I needed customers in every location, thus the question as worded is wrong. Thus I will give a significant amount of leeway in answers. For example not noticing the problem is fine; as is providing a counter example. In the latter case I will have to give you credit for the next part. ■

(e) *(3 points)* Prove that this game can be solved by iterated deletion of dominated strategies.

Proof. In stage one as long as neither 1 nor L are the median location, both will be eliminated by the proof above. This means that the set of locations that is consistent with the common knowledge of rationality is $\{2, 3, 4, ..., L-1\}$, and then we can apply the proof in the last section again until we are left with only the median location. ■

5. *(22 points total)* Consider a model of adverse selection. A seller has a car who's worth is $w \in \{5, 20, 50\} = \{w_l, w_m, w_h\}$. The buyer does not know the worth of the car, the buyer only knows that $\Pr(w = w_h) = \Pr(w = w_m) = \frac{2}{5}$, and $\Pr(w = w_l) = \frac{1}{5}$. Their value of buying the car is $3w$. A seller's strategy is $p \geq 0$, and a buyers strategy is a p^+ where they will buy the car if $p \leq p^+$. We assume that p^+ is set at the maximum it can be given what cars are sold. Their utilities are:

$$\pi_{seller}(w, p, p^+) = \begin{cases} p - w & \text{if} \quad p \leq p^+ \\ 0 & \text{if} \quad p > p^+ \end{cases} ; u_{buyer}(w, p, p^+) = \begin{cases} vw - p & \text{if} \quad p \leq p^+ \\ 0 & \text{if} \quad p > p^+ \end{cases} .$$

(a) *(6 points total)* Find the expected value of the car if:

i. *(2 points)* All cars are sold.

$$\begin{aligned} E[vw] &= vE[w] \\ &= 3\left(\frac{2}{5}(50) + \frac{2}{5}(20) + \frac{1}{5}(5)\right) \\ &= 3 * 29 \\ &= 87 \end{aligned}$$

ii. *(2 points)* High worth cars are not sold.

$$\begin{aligned} E[vw|w < w_h] &= 3 * \left(\frac{\frac{2}{5}}{\frac{2}{5}+\frac{1}{5}}(20) + \frac{\frac{1}{5}}{\frac{2}{5}+\frac{1}{5}}(5)\right) \\ &= 3 * 15 = 45 \end{aligned}$$

iii. *(2 points)* Only low worth cars are sold.

$$E[vw|w = w_l] = 3 * 5 = 15$$

(b) *(9 points total)* Find out if there is an equilibrium when:

i. *(3 points)* All cars are sold.

Solution 9 *Yes because* $w_h = 50 < 87$

ii. *(3 points)* High worth cars are not sold.

Solution 10 *Yes because* $w_h = 50 > 45$ *so high worth cars will not be sold and* $w_m = 20 < 45$

iii. *(3 points)* Only low worth cars are sold.

Solution 11 *Yes because* $w_m = 20 > 15$ *so high and medium worth cars will not be sold.*

(c) *(3 points)* Consider a real world situation where currently only the low worth cars are being sold. Assuming that a trustworthy third party wanted to intervene in the market to get higher worth goods being sold, what problems would they face?

Solution 12 *This is an open ended question, so all relevant answers might get partial credit. The problem that I find most insurmountable is information. How are they going to learn the distribution of worths of cars that are not sold? If the market price is 15 we will know that there are many cars that are not being sold, but what their worth is will not be easy to observe.*

(d) *(4 points)* In this model we always assume $w_l > 0$. This is frankly unrealistic, as you should know there are some cars that will cost more to maintain and fix up than they could ever benefit you. Allowing $p < 0$ but changing nothing else about this model, what would happen if $w_l < 0$? You should consider the case when $|w_l|$ is small and $|w_l|$ is large.

Solution 13 *There are two possible answers for the case where* $|w_l|$ *is small. If we multiply* w_l *by a constant* $v > 1$ *this will mean that the value to customers of these vehicles will be below the worth and the market will truly collapse with no cars sold.*

On the other hand in the original model the value function needed to satisfy $v(w) > w$ *and in this case there will be trade—basically junk people will come by and you'll pay them to haul your piece of trash to the dump.*

If $|w_l|$ *is large this will cause all of the other equilibria to fail.*