ECON 439
Final: Extensive Form Games

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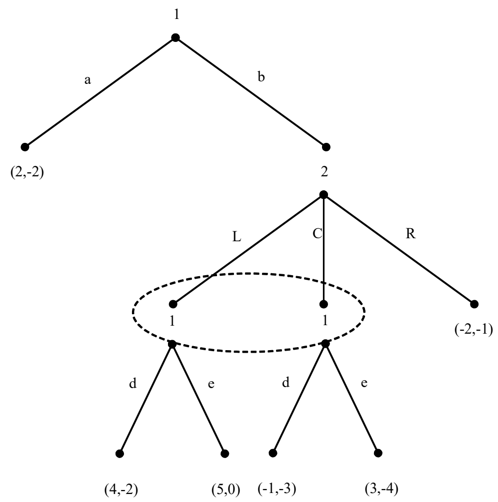
This exam will start at or after 17:45 and must be turned into Moodle by 20:15

Points will only be given for work shown.

1. *(14 points)* Please write the following statement on the first page of your answers and below it write your full name, student ID, and sign it.

   I promise that I have neither given nor received aid from another person while taking this final exam. The following answers are in my own words and based on my own work using class notes, books, and online resources but no human assistance.

2. *(28 points total)* Consider the following extensive form game:



   (a) *(4 points)* Find the strategies of both players.

$$
\begin{aligned}
S_1 &= \{a,b\} \times \{d,e\} \\
&= \{(a,d), (a,e), (b,d), (b,e)\} \\
S_2 &= \{L,C,R\}
\end{aligned}
$$

   (b) *(8 points)* Convert this into a strategic or normal form game, find the best responses and pure strategy Nash equilibria.

	L	C	R
(a,d)	$(2,-2)^2$	$(2,-2)^2$	$(2,-2)^{12}$
(a,e)	$(2,-2)^2$	$(2,-2)^2$	$(2,-2)^{12}$
(b,d)	$(4,-2)$	$(-1,-3)$	$(-2,-1)^2$
(b,e)	$(5,0)^{12}$	$(3,-4)^1$	$(-2,-1)$

(c) *(4 points)* Treating this as a sequential game where player 1 knows whether player 2 played L or C find the subgame perfect equilibrium.

Solution 1 $BR_1(b,L)=e$, $BR_1(b,C)=e$, $BR_2(b)=L$, $BR_1(\emptyset)=b$

(d) *(4 points)* Convert this into a strategy of the game as given, is it a weak sequential equilibrium? If it is state the range of beliefs at the information set $\{L,C\}$ that support it as an equilibrium.

Solution 2 *This strategy would be* $(b,e)(L)$ *and it is a weak sequential equilibrium, to be precise* $\Pr(L|L,C)=1$ *but it can be anything because* e *dominates* d*.*

(e) *(4 points)* Find a subgame perfect equilibrium that is not a weak sequential equilibrium and explain why it is subgame perfect. Hint: It must be a Nash equilibrium.

Solution 3 *To answer this I will find the Nash equilibria of the subgame* b*:*

	L	C	R
d	$(2,-2)$	$(-1,-3)$	$(-2,-1)^{12}$
e	$(3,0)^{12}$	$(1,-4)$	$(-2,-1)^{1}$

thus (d,R) *is a Nash equilibria, and if* (d,R) *then the best response of player 1 is* a*. Thus the strategy* $(a,d)(R)$ *is a subgame perfect equilibrium but not a weak sequential equilibrium because for all* $\Pr(L|L,C)$ e *is the best response.*

(f) *(4 points)* Find a Nash equilibrium that is neither subgame perfect nor a weak sequential equilibrium. Identify who makes an empty threat in this equilibrium, does it help the player who makes it?

Solution 4 *There is only one more Nash equilibrium, which is* $(a,e)(R)$*. Since* R *is not the best response to* e *this is not a subgame perfect equilibrium, and it is not weak sequential equilibrium because player 2 must believe* $\Pr(e|e,d)=1$ *and* R *is not the best response to that.*

The empty threat is made by player 2, he states he will play R *even though if player 1 plays* b *he will not play* R*. Amusingly enough it does not help him to make this empty threat, but he still makes it.*

3. *(26 points)* Consider the following economy, throughout this question you may assume people always vote as if they were pivotal.

1	2	3
D	C	E
C	E	B
E	B	A
A	D	D
B	A	C

(a) *(3 points)* An option is Pareto efficient if no other is better than it for all people. Identify the Pareto efficient options and for those that are not Pareto efficient state something that Pareto dominates it.

Solution 5 $\{C, D, E\}$ *are Pareto efficient,* C *pareto dominates* $\{A, B\}$.

(b) *(5 points)* Fill out the empty squares in the following table indicating which option will win a majority of votes of each pair. In row X and column Y first indicate which gets the majority (X or Y) and then who will vote for the winning option.

vs.	B	C	D	E
A	$B\,(2,3)$	$C\,(1,2)$	$D\,(1,2)$	$E\,(1,2,3)$
B	XX	$C\,(1,2)$	$B\,(2,3)$	$E\,(1,2,3)$
C	XX	XX	$D\,(1,3)$	$C\,(1,2)$
D	XX	XX	XX	$E\,(2,3)$

(c) *(3 points)* Find the top cycle of this economy.

Solution 6 *A always looses, so it can not be in the top cycle. B wins against D and A, D wins against C and A, C wins against $\{A, B, E\}$, E wins against $\{A, B, D\}$. Thus the top cycle is (B, D, C, E)*

In the following three parts, consider the agenda (A, B, D, E, C).

(d) *(5 points total)* In the committee model: one first votes to accept the first option in the agenda or reject it and move on to the second option. This process is repeated until the last option is accepted if no other option has been. Find the winner with the agenda (A, B, D, E, C) in this model.

i. *(1 point)* First will C be accepted or will we reject C and accept E?

Solution 7 *C will be accepted because* $(1, 2)$ *prefer it to E.*

ii. *(1 point)* Then will we accept D or reject it and move on?

Solution 8 *D will be accepted because* $(1, 3)$ *prefer it to C.*

iii. *(1 point)* Will we accept B or reject it and move on?

Solution 9 *B will be accepted because* $(2, 3)$ *prefer B to D.*

iv. *(1 point)* Will we accept A or reject it and move on?

Solution 10 *B will be accepted because* $(2, 3)$ *prefer B to A.*

v. *(1 point)* Which option will be selected?

Solution 11 B

(e) *(6 points)* In the incumbency model: first one votes over the first two options in the agenda. The one that gets a majority of the votes becomes the incumbent, and moves on to compete against the third option. This process repeats until all options are voted on. Find the winner with the agenda (A, B, D, E, C) in this model.

i. *(1 point)* In the 4th round it will be C versus X for $X \in \{A, B, D, E\}$. State who will in a contest of E versus X for all X.

Solution 12 *From the table above:*

vs.	B	C	D	E
C	$C\,(1,2)$	$C\,(1,2)$	$D\,(1,3)$	$C\,(1,2)$

thus a vote for anything but D will actually result in C winning.

ii. *(2 points)* In the 3rd round the contest will be E versus X for $X \in \{A, B, D\}$. Given the outcomes you found above, which option will win each contest?

Solution 13 *Above we found that a vote for A, B, or E is actually a vote for C, thus the real contest is between C and D and since $(1,3)$ prefer D they will vote for D in the (E, D) contest, in the other contests it does not matter how they vote since in the next round C will win.*

iii. *(1 point)* In the 2nd round the contest will be D versus X for $X \in \{A, B\}$. Given the outcomes you found above, which option will win each contest?

Solution 14 *If they vote for A or B then in the final round C will win, thus $(1,3)$ will vote for D in all of these contests.*

iv. *(1 point)* In the first round the contest will be A versus B. Given the outcomes you found above, which option will win?

Solution 15 *It does not matter, any vote for either of these options will end up being a vote for D.*

v. *(1 point)* Which option will be selected?

Solution 16 D

(f) *(4 points)* Consider the incumbency model where agents vote naively. I.e. they simply vote for the option in front of them that they like the most. Find which option will be selected with this type of voting and identify at least one voter who would like to change their vote and vote strategically.

Solution 17 *A versus B, B wins. B verus D, B wins. B versus E, E wins, E verus C, C wins.*

Thus the outcome is C when the strategic outcome is D. Players $(1,3)$ prefer D to C, if player 3 had voted for D in B verus D then D would be the outcome but they voted for B and will regret it.

4. *(32 points total)* Consider a Spence signalling model, where agents use education to signal their ability. Their are two types of workers, h and l with $\Pr(h) = \frac{2}{3}$. Education does not affect productivity thus regardless of their level of education a high productivity worker has a value of $\pi_h =$

36 to a firm and a low productivity worker has a value of $\pi_l = 6$ to a firm. Education costs less for the high types, the marginal cost of a year of education is $c_h = 3$ for a high type, for a low type it is $c_l = 10$. Firms are Bertrand competitors thus if a worker goes to school for e years they will get $w(e) = \beta(e)\pi_h + (1-\beta(e))\pi_l$ where $\beta(e) = \Pr(h|e)$ is the common beliefs of the firms that a workers is a high type given e years of education. For $x \in \{h, l\}$ the utility of a worker is $u_x(w, e) = w - c_x e$. Without loss of generality we can characterize an equilibrium as a (w_h, e_h, w_l, e_l) where a high type worker chooses to go to school for e_h years to earn a wage of w_h and a low type worker chooses to go for e_l and earn w_l.

(a) *(8 points)* Write down the four constraints that summarize a worker's optimal behavior for a given (w_h, e_h, w_l, e_l).

$$\begin{aligned} IR_h &: w_h - 3e_h \geq 6 \\ IR_l &: w_l - 10e_l \geq 6 \\ IC_h &: w_h - 3e_h \geq w_l - 3e_l \\ IC_l &: w_h - 10e_h \leq w_l - 10e_l \end{aligned}$$

(b) *(4 points)* Which of these four constraints will never be used when analyzing equilibria? Prove your statement.

Solution 18 *You will never use IR_h because by IC_h we know that:*

$$w_h - 3e_h \geq w_l - 3e_l$$

and obviously:

$$w_l - 3e_l \geq w_l - 10e_l$$

and by IR_l we know that

$$w_l - 10e_l \geq 6$$

thus we derive:

$$w_h - 3e_h \geq w_l - 3e_l \geq w_l - 10e_l \geq \pi_l$$

and this is IR_h. Notice this also has the cool implication that higher types always get a weakly higher payoff than low types.

(c) *(8 points total)* For each of the following state whether it can be a pure strategy weak sequential equilibrium or not. If it can be prove this, if it can not explain why.

i. *(2 points)* $(w_h, e_h, w_l, e_l) = (26, 1, 26, 1)$

Solution 19

$$26 = \frac{2}{3}36 + \frac{1}{3}6$$

so this can be a pooling equilibrium if the low types are willing to go to school for one year in order to get a wage of 26, this is true because

$$26 - 10 * 1 = 16 > 6$$

since $\pi_l = 6$.

ii. *(2 points)* $(w_h, e_h, w_l, e_l) = (36, 11, 6, 0)$

Solution 20 $(6, 0)$ *is fine in a seperating equilibrium, so the only question is whether the high types are willing to go to school for* 11 *years to get a wage of* 36.

$$36 - 3 * 11 = 3 < 6$$

so this is not an equilibrium because e_h *is too high.*

iii. *(2 points)* $(w_h, e_h, w_l, e_l) = (30, 6, 6, 0)$

Solution 21 *Since* $e_h \neq e_l$ *this is a seperating equilibrium, but then* $w_h = \pi_h = 36$*, not* 30.

iv. *(2 points)* $(w_h, e_h, w_l, e_l) = (36, 4, 8, 0)$

Solution 22 *This is again a seperating equilibrium, but* $w_l = 8 > \pi_l$ *so it is not an equilibrium.*

(d) *(6 points)* Characterize the equilibria where $e_h \neq e_l$. What is the name of this class of equilibria? Who is signalling in these equilibria and what does the signal do?

Solution 23 *Since* $e_h \neq e_l$ *these are all* seperating *equilibria, where given someone's education level the firms know their type. This means that* $w_h = 36$, $w_l = 6$, *and individual rationality for the low type tells us that:*

$$6 - 10e_l \geq 6$$

thus $e_l = 0$. *Thus incentive compatability for the high type gives us:*

$$\begin{aligned} 36 - 3e_h &\geq 6 \\ 30 &\geq 3e_h \\ 10 &\geq e_h \; . \end{aligned}$$

Incentive compatability for the low type gives us:

$$\begin{aligned} 36 - 10e_h &\leq 6 \\ 30 &\leq 10e_h \\ 3 &\leq e_h \end{aligned}$$

thus the class of seperating equilria is: $(w_h, e_h, w_l, e_l) = (36, e_h, 6, 0)$ *where* $3 \leq e_h \leq 10$.

The high types are signalling in these equilibria, and the signal reveals their true quality.

(e) *(6 points)* Characterize the equilibria where $e_h = e_l$. What is the name of this class of equilibria? Who is signalling in these equilibria and what does the signal do? What is the one equilibrium in this class where no one is signalling?

Solution 24 *Since $e_h = e_l$ $w_h = w_l$ since firms can not tell the workers apart. Thus*

$$w_h = w_l = \lambda \pi_h + (1 - \lambda) \pi_l = \frac{2}{3} 36 + \frac{1}{3} 6 = 26$$

these are pooling equilibria *and we only have one constraint to check— IR_l or:*

$$\begin{aligned} 26 - 10 e_p &\geq 6 \\ 20 &\geq e_p \\ 2 &\geq e_p \end{aligned}$$

thus these equilibria are $(w_h, e_h, w_l, e_l) = (26, e_p, 26, e_p)$ where $0 \leq e_p \leq 2$.

If $e_p > 0$ then everyone is signalling. They are all trying to indicate they are not low quality workers, and specifically low types want to hide the fact that they are low types.

The only equilibrium without signalling is when $e_p = 0$.