ECON 439
Midterm: Normal Form Games

Kevin Hasker

This exam will start at about 10:30 and will end around 12:10

Points will only be given for work shown.

1. *(14 points)* **Honor Statement:** Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test. I will also not offer assistance to others. Finally I will not use a calculator or other electronic aid for calculation during this test.

   Name and Surname: _

   Student ID: _

   Signature:

   _

2. *(8 points)* Consider an n player game where each player has a finite number of strategies. Will there be a Nash equilibrium of this game? Explain you answer and discuss why this information is important.

   Solution 1 *As Nash proved, there is always a Nash equilibrium in a game where all players have a finite number of strategies. Notice that (of course) these equilibria might be in mixed strategies—i.e. people will want to be unpredictable.*

   This is vitally important since most interactions can be approximated as having a finite number of strategies, and we know for a fact there must be a Nash equilibrium in them. In other words we will never be looking for an empty set.

3. *(22 points total)* Consider a game where there are two states of the world, ω_1 and ω_2. Player 1 does not know the state of the world and believes $\Pr(\omega_1) = \frac{1}{2}$. Player 2 knows the state of the world. Player 1 has the strategies $\{A, B, C\}$ and player 2 has the strategies $\{\alpha, \beta, \gamma\}$ if the state is ω_1 and $\{\delta, \tau, \chi\}$ if the state is ω_2. The Normal form games are:

ω_1	α	β	γ
A	$0;3$	$0;4$	$7;7^{12}$
B	$4;1^1$	$4;2$	$4;3^2$
C	$0;1$	$5;5^{12}$	$0;4$

ω_2	δ	τ	χ
A	$7;5^{12}$	$0;4$	$0;1$
B	$4;2$	$4;5^{12}$	$4;3$
C	$0;2$	$0;3$	$5;5^{12}$

	$BR_2(A)$ (γ,δ)	$BR_2(B)$ (γ,τ)	$BR_2(C)$ (β,χ)
A	7^1	$\frac{7}{2}$	0
B	4	4^1	4
C	0	0	5^1

ω_1	α	β	γ
A	$6;1$	$6;5^2$	$6;4^1$
B	$0;3$	$8;5^{12}$	$0;2$
C	$10;7^{12}$	$0;3$	$0;4$

ω_2	δ	τ	χ
A	$6;2$	$6;1$	$6;3^{12}$
B	$0;4$	$8;7^{12}$	$0;3$
C	$10;3^{12}$	$0;1$	$0;2$

	$BR_2(A)$ (β,χ)	$BR_2(B)$ (β,τ)	$BR_2(C$ (α,δ)
A	6^1	6	6
B	4	8^1	0
C	0	0	10^1

	ω_1		
	α	β	γ
A	$0;2$	$0;3$	$7;5^{12}$
B	$4;7^{2}$	$4;3^{1}$	$4;4$
C	$5;3^{12}$	$0;1$	$0;2$

			ω_2
	δ	τ	χ
A	$0;1$	$0;4$	$7;5^{12}$
B	$4;5^{12}$	$4;4$	$4;1$
C	$0;2$	$5;5^{12}$	$0;3$

	$BR_2(A)$ (γ,χ)	$BR_2(B)$ (α,δ)	$BR_2(C)$ (α,τ)
A	7^{1}	0	0
B	4	4^{1}	4
C	0	$\frac{5}{2}$	5^{1}

	ω_1		
	α	β	γ
A	$0;1$	$0;4$	$10;5^{12}$
B	$8;3^{12}$	$0;2$	$0;1$
C	$6;3$	$6;2^{1}$	$6;5^{2}$

	ω_2		
	δ	τ	χ
A	$0;2$	$0;1$	$10;3^{12}$
B	$8;7^{12}$	$0;4$	$0;3$
C	$6;4$	$6;7^{12}$	$6;3$

	$BR_2(A)$ (γ,χ)	$BR_2(B)$ (α,δ)	$BR_2(C$ (γ,τ)
A	10^{1}	0	5
B	0	8^{1}	0
C	6	6	6^{1}

(a) *(6 points)* Treating the game as if player 1 knows whether the state is ω_1 or ω_2, find the pure strategy best responses of both players in both games. You may use the table to mark your best response but you will loose two points if you do not explain your notation below.

Solution 2 *The best responses are marked in the games above, a 1 (2) in the upper right hand corner indicates this is the best response for player 1 (2).*

(b) *(4 points)* In the game as described, fill out the following table for $x \in \{A, B, C\}$.

Solution 3 *See above, this is constructed for each game, I include what each best response is for clarity.*

(c) *(5 points)* Using the table you just constructed, find the pure strategy Bayesian Nash equilibrium of this game.

Solution 4 *This is an error, in each game as constructed their are three NE, for all $x \in \{A, B, C\}$, $(x, BR_2(x))$ is a Nash equilibrium. If you wrote down one I gave full credit.*

(d) *(7 points)* Compare the Bayesian Nash equilibrium you found to the best responses of the games played under full information (in part *a* of this question.) What is peculiar about this equilibrium? What does it show us about why we can not remove pure strategies from consideration when they are never a weak best response?

Solution 5 *Again, this is an error. There is no strategy that is never a best response. Thus I gave full credit to everyone regardless of whether they answered the question or not.*

4. *(20 points)* About dominance and weak dominance.

(a) *(4 points)* Define a dominated strategy.

Solution 6 *A dominates B if for all $s_{-i} \in S_{-i}$ $u_i(A, s_{-i}) > u_i(B, s_{-i})$. Note the importance of stating it is the **same** strategy **for all** strategies of the others.*

(b) *(4 points)* Explain why we can iterate the concept of dominance, and conclude that as long as players are rational and this is common knowledge no one will use a strategy that does not survive this process.

Solution 7 *Clearly if I know you are rational I can assume you will not play a dominated strategy, and I can delete it. Logically it seems that we should be able to repeat this ad-infinitum, and there is a theorem that proves that we can. To be specific no matter what our process as long as we keep deleting dominated strategies* in the current game *we will always end up with the same set of strategies. This matters because it means that as long as we know everyone is rational we will always arrive at the same conclusion.*

(c) *(4 points)* Define a weakly dominated strategy.

Solution 8 *A weakly dominates B if for all* $s_{-i} \in S_{-i}$ $u_i(A, s_{-i}) \geq u_i(B, s_{-i})$ *and for some* s^*_{-i} $u_i(A, s^*_{-i}) > u_i(B, s^*_{-i})$.

(d) *(4 points)* Explain why if players are rational and this is common knowledge we *can not* safely iteratively remove weakly dominated strategies.

Solution 9 *I just want to say how pleased I was at your answers to this question. Not only did you clearly basically get the point* ***but*** *several of you came up with interesting points that I did not emphasize in class.*

Several noted that since something can be weakly dominated and a best response we can remove a Nash equilibrium using this method. That is fascinating, but not really that big of an issue. (Still worth a significant amount of credit)

The fundamental problem is that the order of deletion can matter. If we remove strategy s^*_{-i} *before B then B may no longer be weakly dominated. Thus two rational people can disagree on what the end process should be, thus it can not be an implication only of common knowledge of rationality.*

(e) *(4 points)* Consider a modified game where every strategy will be played with some small probability $\varepsilon > 0$. Will there be any weakly dominated strategies that are not dominated with this modification?

Solution 10 *One of you got this. I am pleased that one of you did. Let the number of strategies of other players be* $n+1$ $(\#(S_{-i}) = n+1)$

and the number for i be $m+1$ then for any s_i

$$\begin{aligned} U_i(s_i, s_{-i}) &= (1-m\varepsilon)\left[(1-n\varepsilon)u_i(s_i, s_{-i}) + \varepsilon \sum_{\tilde{s}_{-i}\in S_{-i}\backslash s_{-i}} u_i(s_i, \tilde{s}_{-i})\right] \\ &+\varepsilon \sum_{\tilde{s}_i\in S_i\backslash s_i} \left[(1-n\varepsilon)u_i(\tilde{s}_i, s_{-i}) + \varepsilon \sum_{\tilde{s}_{-i}\in S_{-i}\backslash s_{-i}} u_i(\tilde{s}_i, \tilde{s}_{-i})\right] \end{aligned}$$

$$\begin{aligned} U_i(s_i, s_{-i}) &= (1-m\varepsilon)(1-n\varepsilon)u_i(s_i, s_{-i}) + \varepsilon \sum_{\tilde{s}_i\in S_i\backslash s_i} (1-n\varepsilon)u_i(\tilde{s}_i, s_{-i}) \\ &+ (1-m\varepsilon)\varepsilon \sum_{\tilde{s}_{-i}\in S_{-i}\backslash s_{-i}} u_i(s_i, \tilde{s}_{-i}) \\ &+\varepsilon^2 \sum_{\tilde{s}_i\in S_i\backslash s_i} \sum_{\bar{s}_{-i}\in S_{-i}\backslash s_{-i}} u_i(\tilde{s}_i, \bar{s}_{-i}) \end{aligned}$$

$$\begin{aligned} U_i(s_i, s_{-i}) &= (1-(m+1)\varepsilon)(1-(n+1)\varepsilon)u_i(s_i, s_{-i}) + \varepsilon \sum_{\tilde{s}_i\in S_i\backslash s_i} (1-(n+1)\varepsilon)u_i(\tilde{s}_i, s_{-i}) \\ &+ (1-m\varepsilon)\varepsilon \sum_{\tilde{s}_{-i}\in S_{-i}} u_i(s_i, \tilde{s}_{-i}) \\ &+\varepsilon^2 \sum_{\tilde{s}_i\in S_i} \sum_{\bar{s}_{-i}\in S_{-i}} u_i(\tilde{s}_i, \bar{s}_{-i}) \end{aligned}$$

$$\begin{aligned} U_i(s_i, s_{-i}) &= (1-(m+1)\varepsilon)(1-(n+1)\varepsilon)u_i(s_i, s_{-i}) + (1-m\varepsilon)\varepsilon \sum_{\tilde{s}_{-i}\in S_{-i}} u_i(s_i, \tilde{s}_{-i}) \\ &+\varepsilon \sum_{\tilde{s}_i\in S_i\backslash s_i} (1-(n+1)\varepsilon)u_i(\tilde{s}_i, s_{-i}) \\ &+\varepsilon^2 \sum_{\tilde{s}_i\in S_i} \sum_{\bar{s}_{-i}\in S_{-i}} u_i(\tilde{s}_i, \bar{s}_{-i}) \end{aligned}$$

We note that the first line is now a function of s_i, and the second is a constant that only depends on s_{-i}, the third is a constant that depends on nothing.

Looking at the first line we notice that if A weakly dominates B then in the first term $u_i(A, s_{-i}) \geq u_i(B, s_{-i})$, and in the second $\sum_{\tilde{s}_{-i}\in S_{-i}} u_i(A, \tilde{s}_{-i}) > \sum_{\tilde{s}_{-i}\in S_{-i}} u_i(B, \tilde{s}_{-i})$ by the definition of A weakly dominates B, thus $U_i(A, s_{-i}) > U_i(B, s_{-i})$ in this modified game.

And of course I did not require this degree of detail in the answer. I *am merely being thorough in my answers.*

b	χ	*NE*	$p_0\,(p_1)$	p_0	p_1	p_2
4	3	$\{0\}\times\{2\}$	$\frac{3}{5}-\frac{2}{5}p_1$	$\frac{4}{7}$	$\frac{1}{14}$	$\frac{5}{14}$
8	6	$\{0\}\times\{2\}$	$\frac{3}{5}-\frac{2}{5}p_1$	$\frac{4}{7}$	$\frac{1}{14}$	$\frac{5}{14}$
6	9	$\{0\}\times\{1,2\}$	$\frac{3}{4}-\frac{1}{4}p_1$	$\frac{8}{11}$	$\frac{1}{11}$	$\frac{2}{11}$
2	3	$\{0\}\times\{1,2\}$	$\frac{3}{4}-\frac{1}{4}p_1$	$\frac{8}{11}$	$\frac{1}{11}$	$\frac{2}{11}$

5. *(30 points)* Consider a war of attrition where they can fight for up to two periods. Each player ($i\in\{1,2\}$) can choose $t_i\in\{0,1,2\}$. They have a symmetric payoff function which is:

$$u_i\left(t_i,t_j\right)=\begin{cases} b-\chi t_j & \text{if} \quad t_i>t_j \\ \frac{b}{2}-\chi t_j & \text{if} \quad t_i=t_j \\ -\chi t_i & \text{if} \quad t_i<t_j \end{cases}$$

for $i\in\{1,2\}$, $j\neq i$.

(a) *(9 points)* Convert this to a standard normal/strategic form game, drawing the table below.

Solution 11 *I was shocked that only about half of you could do this, and of those who could at least write the basic structure maybe 10% got the payoffs wrong.*

	$t_2=0$	$t_2=1$	$t_2=2$
$t_1=0$	$\frac{b}{2};\frac{b}{2}$	$0;b^{2(1)}$	$0;b^{21}$
$t_1=1$	$b;0^{1(2)}$	$\frac{b}{2}-\chi;\frac{b}{2}-\chi$	$-\chi;b-\chi^{(2)}$
$t_1=2$	$b;0^{12}$	$b-\chi;-\chi^{(1)}$	$\frac{b}{2}-\chi 2;\frac{b}{2}-\chi 2$

Several of you decided to just write down an arbitrary simple game and answer the question for that game. While if you just wrote down the wrong payoffs I (generally speaking) graded the rest of the question given this writing down a completely new game resulted in zero credit.

(b) *(6 points)* Find the pure strategy best responses of both players, you may use the table you just drew but you will loose two points if you do not explain your notation below.

Solution 12 *I use a* 1 *in the upper right hand corner if it is a best response for player 1, a* 2 *if it is a best response for player* 2*. I have used a parentheses when the answer depends on* $\{b,\chi\}$*. If* $b>\chi$ *then* $t_i=2$ *is a best response to* $t_j=1$*, otherwise* $t_i=0$ *is the best response.*

(c) *(6 points)* Find the pure strategy Nash equilibria, explain why they are Nash equilibria.

Solution 13 *If $b > \chi$ they are $(0,2)$ and $(2,0)$, otherwise $(0,1)$ and $(1,0)$ are also NE. They are Nash equilibria because this is where best responses intersect, or they are both best responding simultaneously. Or (as one put it) given what the other is doing they have no incentive to change their strategy. That is a better answer than mine, because it does not require you know what a best response is.*

(d) *(9 points)* Note that all the pure strategy equilibria are asymmetric, or $t_i^* \neq t_j^*$. Find the symmetric equilibrium, heavy partial credit will be given for work towards the correct answer.

Solution 14 *So I am pissed that so many of you could not construct the game? I was so excited at how many of you realized this must be a mixed strategy equilibrium, and then many of you actually found it? I note that in this equilibrium you have to solve for p_0 and p_1—two variables, while in a standard mixed strategy equilibrium you have to solve for p and q—two variables.*

If there is no symmetric pure strategy equilibrium since I have insisted there is a symmetric equilibrium it must be in mixed strategies. Let $p_x = \Pr(t_i = x)$ for $x \in \{0,1,2\}$. Then let $p_2 = 1 - p_1 - p_0$ for simplicity and our three expected utilities are:

$$
\begin{aligned}
u_1(0,p) &= p_0\frac{b}{2} + p_1(0) + (1-p_1-p_0)\,0 = \frac{1}{2}bp_0 \\
u_1(1,p) &= p_0(b) + p_1\left(\frac{b}{2} - \chi\right) + (1-p_1-p_0)(-\chi) = bp_0 - \chi + \frac{1}{2}bp_1 + \chi p_0 \\
u_1(2,p) &= p_0(b) + p_1(b) + (1-p_1-p_0)\left(\frac{b}{2} - 2\chi\right) = \frac{1}{2}b - 2\chi + \frac{1}{2}bp_0 + \frac{1}{2}bp_1 + 2\chi p_0 + 2\chi p_1
\end{aligned}
$$

At this point let me let $\frac{b}{\chi} = c$ or $\chi = bc$ then:

$$
\begin{aligned}
u_1(0,p) &= \frac{1}{2}bp_0 \\
u_1(1,p) &= bp_0 - (bc) + \frac{1}{2}bp_1 + (bc)\,p_0 \\
u_1(2,p) &= \frac{1}{2}b - 2(bc) + \frac{1}{2}bp_0 + \frac{1}{2}bp_1 + 2(bc)\,p_0 + 2(bc)\,p_1
\end{aligned}
$$

to find p_0 I set

$$
\begin{aligned}
u_1(0,p) &= u_1(1,p) \\
\frac{1}{2}bp_0 &= bp_0 - (bc) + \frac{1}{2}bp_1 + (bc)\,p_0 \\
p_0 &= \frac{2c - p_1}{2c+1}
\end{aligned}
$$

to find p_1 *I set:*

$$\begin{aligned}
u_1(0,p) &= u_1(2,p) \\
\frac{1}{2}b\left(\frac{2c-p_1}{2c+1}\right) &= \frac{1}{2}b-2(bc)+\frac{1}{2}b\left(\frac{2c-p_1}{2c+1}\right)+\frac{1}{2}bp_1+2(bc)\left(\frac{2c-p_1}{2c+1}\right)+2(bc)p_1 \\
\frac{1}{2}b\left(\frac{2c-p_1}{2c+1}\right) &= \frac{1}{2}\frac{b}{2c+1}\left(8p_1c^2+2p_1c+1\right) \\
2c-p_1 &= 8p_1c^2+2p_1c+1 \\
2c-1 &= 8p_1c^2+2p_1c+p_1 \\
2c-1 &= \left(8c^2+2c+1\right)p_1 \\
p_1 &= \frac{2c-1}{2c+8c^2+1} \\
p_0 &= \frac{2c-\left(\frac{2c-1}{2c+8c^2+1}\right)}{2c+1} = \frac{8c^2-2c+1}{8c^2+2c+1} \\
p_2 &= 1-\frac{2c-1}{2c+8c^2+1}-\frac{8c^2-2c+1}{8c^2+2c+1} \\
&= \frac{2c+1}{8c^2+2c+1}
\end{aligned}$$

as it is a quadratic relatively simple solutions only occurred if $p \in \left\{\frac{3}{4},\frac{3}{2}\right\}$. *Notice that the sensitive point is* p_1, *if* $c < \frac{1}{2}$ *then* $p_1 < 0$ *and one has to resolve the problem assuming that* $p_1 = 0$. *Obviously I could not have this. But just to show you how if* $p_1 = 0$ *then*

$$\begin{aligned}
u_1(1,p) &< u_1(0,p) \\
u_1(0,p) &= u_1(2,p)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}bp_0 &= \frac{1}{2}b-2(bc)+\frac{1}{2}bp_0+\frac{1}{2}b(0)+2(bc)p_0+2(bc)(0) \\
\frac{1}{2}bp_0 &= \frac{1}{2}b(p_0-4c+4cp_0+1) \\
p_0 &= \frac{1}{4c}(4c-1)
\end{aligned}$$

to verify we are correct we need to make sure $u_1(0,p^*) > u_1(1,p^*)$

$$\begin{aligned}
u_1(0,p^*) &= \frac{1}{2}b\left(\frac{1}{4c}(4c-1)\right) = \frac{1}{8}\frac{b}{c}(4c-1) \\
u_1(1,p^*) &= b\left(\frac{1}{4c}(4c-1)\right)-(bc)+\frac{1}{2}b(0)+(bc)\left(\frac{1}{4c}(4c-1)\right) \\
&= \frac{1}{4}\frac{b}{c}(3c-1)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{8}\frac{b}{c}(4c-1) &> \frac{1}{4}\frac{b}{c}(3c-1) \\
4c-1 &> 6c-2 \\
1 &> 2c \\
\frac{1}{2} &> c
\end{aligned}$$

6. *(6 points)* Consider a Hotelling location model. There are L locations, with c_l ($l \in \{1,2,3,...,L\}$) customers at each location, $c_l > 0$. These customers will go to firm a if $|l_a - l| < |l_b - l|$, to firm b if $|l_a - l| > |l_b - l|$ and otherwise half will go to each firm. Firms choose their location ($l_x \in \{1,2,3,...,L\}$ for $x \in \{a,b\}$) to maximize their number of customers. Prove *business stealing* or that $BR_a(l_b) \in \{l_b - 1, l_b, l_b + 1\}$.

Solution 15 *When thinking about my answer to this question I realized that it is quite simple to derive the demand curve, and that makes it much clearer. From the description we see that:*

$$D_a(l_a, l_b) = \sum_{l:d(l_a,l)<d(l_b,l)} c_l + \frac{c_l}{2} 1_{d(l_a,l)=d(l_b,l)}$$

where 1_x *is the indicator function,* $1_x = 1$ *if* x *is true and* 0 *otherwise. If* $l_a < l_b$ *then* $d(l_a,l) = d(l_b,l)$ *if* $l - l_a = l_b - l$*, or* $l = \frac{1}{2}l_a + \frac{1}{2}l_b$ *thus*

$$D_a(l_a, l_b) = \sum_{l<\frac{1}{2}l_a+\frac{1}{2}l_b} c_l + \frac{c_{l_a+\frac{1}{2}l_b}}{2} 1_{\frac{1}{2}l_a+\frac{1}{2}l_b \in \{1,2,3,...,L\}} \text{ if } l_a < l_b$$

Likewise if $l_a > l_b$ *then*

$$D_a(l_a, l_b) = \sum_{l>\frac{1}{2}l_a+\frac{1}{2}l_b} c_l + \frac{c_{l_a+\frac{1}{2}l_b}}{2} 1_{\frac{1}{2}l_a+\frac{1}{2}l_b \in \{1,2,3,...,L\}} \text{ if } l_a > l_b$$

and clearly if $l_a = l_b$ $D_a(l_a, l_b) = \frac{1}{2}\sum_l c_l$ *thus:*

$$D_a(l_a, l_b) = \begin{cases} \sum_{l<\frac{1}{2}l_a+\frac{1}{2}l_b} c_l + \frac{c_{\frac{1}{2}l_a+\frac{1}{2}l_b}}{2} 1_{\frac{1}{2}l_a+\frac{1}{2}l_b \in \{1,2,3,...,L\}} & \text{if } \quad l_a < l_b \\ \frac{1}{2}\sum_l c_l & \text{if } \quad l_a = l_b \\ \sum_{l>\frac{1}{2}l_a+\frac{1}{2}l_b} c_l + \frac{c_{\frac{1}{2}l_a+\frac{1}{2}l_b}}{2} 1_{\frac{1}{2}l_a+\frac{1}{2}l_b \in \{1,2,3,...,L\}} & \text{if } \quad l_a > l_b \end{cases}$$

given this we can then easily answer the question. If $1 \leq l_a < l_b - 1$ *then*

$$D_a(l_a+1, l_b) - D_a(l_a, l_b) = \frac{c_{\frac{1}{2}l_a+\frac{1}{2}l_b}}{2} 1_{\frac{1}{2}l_a+\frac{1}{2}l_b \in \{1,2,3,...,L\}} + \frac{c_{\frac{1}{2}l_a+\frac{1}{2}+\frac{1}{2}l_b}}{2} 1_{\frac{1}{2}l_a+\frac{1}{2}+\frac{1}{2}l_b \in \{1,2,3,...,L\}}$$

where we notice that either $\frac{1}{2}l_a + \frac{1}{2}l_b \in \{1,2,3,...,L\}$ *or* $\frac{1}{2}l_a + \frac{1}{2} + \frac{1}{2}l_b \in \{1,2,3,...,L\}$*. Either way it is strictly positive, thus the demand is strictly increasing in* l_a*.*

Likewise if $l_b + 1 < l_a \leq L$ *then*

$$D_a(l_a - 1, l_b) - D_a(l_a, l_b) = \frac{c_{\frac{1}{2}l_a - \frac{1}{2} + \frac{1}{2}l_b}}{2} 1_{\frac{1}{2}l_a - \frac{1}{2} + \frac{1}{2}l_b \in \{1,2,3,\ldots,L\}} + \frac{c_{\frac{1}{2}l_a + \frac{1}{2}l_b}}{2} 1_{\frac{1}{2}l_a + \frac{1}{2}l_b \in \{1,2,3,\ldots,L\}}$$

which is strictly decreasing in l_a*. Thus* $BR_a(l_b) \in \{l_b - 1, l_b, l_b + 1\}$*. To be precise:*

$$D_a(BR_a(l_b), l_b) = \begin{cases} \sum\limits_{l<l_b} c_l & \text{if} \quad l_a = l_b - 1 \\ \frac{1}{2}\sum\limits_{l} c_l & \text{if} \quad l_a = l_b \\ \sum\limits_{l>l_b} c_l & \text{if} \quad l_a = l_b + 1 \end{cases}.$$

This also makes it clear that if $l_b < l_m$ *where* l_m *is* a *median location, then* $\sum\limits_{l>l_b} c_l > \frac{1}{2}\sum\limits_{l} c_l > \sum\limits_{l<l_b} c_l$ *and* $l_a = l_b + 1$ *and if* $l_b > l_m$ *then* $\sum\limits_{l<l_b} c_l > \frac{1}{2}\sum\limits_{l} c_l > \sum\limits_{l>l_b} c_l$ *and* $l_a = l_b + 1$.