ECON 439
Extensive Form Games

Be sure to show your work for all answers, even if the work is simple.

This exam will begin at 9:30 and end at 11:10

1. *(16 points)* **Honor Statement:** Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person. I will also not aid anyone else. Finally I will not use a calculator or other electronic aid for calculation.

   Name and Surname: _
   Student ID: _
   Signature:

   _

2. *(26 points total)* About Equilibrium.

   (a) *(6 points)* Define a Nash equilibrium and explain why (perhaps with an example) a subgame perfect equilibrium is generally a better equilibrium concept than a Nash equilibrium.

   Solution 1 *My preferred definition is that it is a* $\sigma^* \in \times_i \Delta(S_i)$ *such that:*

   i. *There is a* $\beta_i \in \times_{j \neq i} \Delta(S_j)$ *such that* $\sigma_i^* \in \arg\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \beta_i)$

   ii. $u_i(\sigma^*) = u_i(\sigma_i, \beta_i)$

   There are many reasonable alternatives. Giving an example is not a definition.

   The problem with this equilibrium concept is that $u_i(\sigma^*) = u_i(\sigma_i, \beta_i)$ *only pins down what will occur at positive probability events. This gives rise to games like the entry game where the incumbent can threaten to fight if the entrant enters and the entrant will stay out because of this. This is unreasonable expectations, but it's fine because on the equilibrium path the incumbent does nothing.*

   (b) *(2 points)* Give an example of a game where empirically people do not play the subgame perfect equilibrium. (In all examples I can think of the verification is from experiments.)

   Solution 2 *Three examples, in the ultimatum game experimentally the outcome is around 50% share for the second player, while the SPE is 1 lira or zero.*

   In the centipede game people usually start out saying in (continuing the game) and grab the prize somewhere before the end.

   Likewise in the finitely repeated prisoner's dilemma people will start out cooperating.

(c) *(6 points)* Define a *subgame perfect equilibrium* and explain why (perhaps with an example) weak sequential equilibrium is generally a better equilibrium concept than a subgame perfect equilibrium. **NOTE:** Be sure to define a subgame.

Definition 3 *A* subgame *begins with a partial history h such that it is common knowledge h has occurred, and it is the game that follows this h. A subgame perfect equilibrium is a Nash equilibrium in every subgame.*

Subgames are easily broken up, for example in the book they give the entrant in the entry game the option to prepare for a fight with secret actions. This is enough to destroy the subgame. Weak sequential equilibrium (generally speaking) continues to embrace the essence of SPE without being tripped up by such trivial changes.

(d) *(8 points)* Define a *weak sequential equilibrium.*

Solution 4 *A weak sequential equilibrium is an assessment* (σ^*, β^*) *where* σ^* *is an equilibrium strategy and* β^* *are beliefs about other's strategies and nature.*

σ^* *must be sequentially rational given* β^**, or for each i and h,* $\sigma_i^*(h)$ *must be a best response to* $\beta_i^*(h)$

And β_i^* *must be consistent with* σ_{-i}^* *and use Bayes rule whenever possible.*

(e) *(4 points)* What is the general problem with Nash equilibrium that means we need refinements of it like subgame perfect equilibrium and weak sequential equilibrium?

Solution 5 *You might have your own answer here (not that any of you did), but I see it as "the problem of zero," after a zero probability event any beliefs and actions are reasonable so you can do and believe whatever you want.*

3. *(36 points total)*Consider a group of three people $\{1, 2, 3\}$ who has to choose among five alternatives: $\{A, B, C, D, E\}$. Their preferences are as follows:

1	E	3
B	A	D
A	C	E
D	D	B
E	E	C
C	B	A

	B	C	D	E
A	$B\{1,3\}$	$A\{1,2\}$	$A\{1,2\}$	$A\{1,2\}$
B		$B\{1,3\}$	$D\{2,3\}$	$E\{2,3\}$
C			$D\{1,3\}$	$E\{1,3\}$
D				$D\{1,2,3\}$

(E, C, B, A, D)

$Committee - E$

$Statusquo - A$

1	2	3
C	D	A
D	E	B
A	A	C
B	B	E
E	C	D

	B	C	D	E
A	$A\{1,2,3\}$	$A\{2,3\}$	$D\{1,2\}$	$A\{1,3\}$
B		$B\{2,3\}$	$D\{1,2\}$	$B\{1,3\}$
C			$C\{1,3\}$	$C\{1,3\}$
D				$D\{1,2\}$

(B, E, C, D, A)

$Committee - B$

$Statusquo - D$

1	2	3
D	A	B
A	C	E
B	B	D
E	E	C
C	D	A

	B	C	D	E
A	$A\{1,2\}$	$A\{1,2\}$	$D\{1,3\}$	$A\{1,2\}$
B		$B\{1,3\}$	$B\{2,3\}$	$B\{1,2,3\}$
C			$D\{1,3\}$	$E\{1,3\}$
D				$E\{2,3\}$

(D, A, B, E)

(E, C, D, A, B)

1	2	3
E	B	A
B	D	C
A	A	E
C	C	D
D	E	B

	B	C	D	E
A	$B\{1,2\}$	$A\{1,2,3\}$	$A\{1,3\}$	$A\{2,3\}$
B		$B\{1,2\}$	$B\{1,2\}$	$E\{1,3\}$
C			$C\{1,3\}$	$C\{2,3\}$
D				$E\{1,3\}$

(E, B, A, C)

(C, D, E, B, A)

1	2	3
A	B	C
B	E	D
C	C	A
D	D	E
E	A	B

	B	C	D	E
A	$A\{1,3\}$	$C\{2,3\}$	$D\{2,3\}$	$A\{1,3\}$
B		$B\{1,2\}$	$B\{1,2\}$	$B\{1,2\}$
C			$C\{1,2,3\}$	$C\{1,3\}$
D				$D\{1,3\}$

(C, D, E, A, B, C)

For some reason they have agreed on the agenda—order of considering options—of (D, E, A, B, C), but have not agreed whether to use the standard committee model or the status quo model to select the outcome. First, let's do some preliminary analysis.

(a) *(2 points)* What does it mean if a person's vote is *pivotal*?

Solution 6 *It means that if this voter changes their vote then the outcome of the election will change.*

(b) *(4 points)* Prove that any $X \in \{A, B, C, D, E\}$ can be the subgame perfect equilibrium with either the standard committee model or the status quo model using the agenda above. **Note:** you may not assume people always vote as if they were pivotal.

Solution 7 *Assume that everyone believes that the other two people will always vote in favor of X when it appears in the agenda. Then there is probability zero that their action will affect the outcome. But this means they can do whatever they want, so assume that they will always vote in favor of X when it appears in the agenda, either rejecting previous options in the committee model, or simply making it the status quo in the status quo model.*

(c) *(10 points)* Fill out the following table with which of the two options a majority of the people prefer—after the option write who prefers it. For example if people 1 and 3 prefer B to D then you would write $B\{1,3\}$ in the second row and third column of this table.

Solution 8 *See the solutions for each variation of the problem above.*

(d) *(5 points)* Define the *top cycle,* and then find it for this example.

Solution 9 *The* top cycle *can be defined two ways:*

i. We say that x beats y if a majority prefer x to y, and that x indirectly beats y if there is a finite sequence of options where x beats the first, which beats the second and so on until the last one beats y.
Then the top cycle is the set of options that indirectly beat all other options.

ii. Alternatively, the top cycle is the smallest set of options such that nothing outside the set beats any member of the set.

I will answer the second part for this economy:

1	2	3
A	B	C
B	E	D
C	C	A
D	D	E
E	A	B

(D, E, A, B, C)

	B	C	D	E
A	$A\{1,3\}$	$C\{2,3\}$	$D\{2,3\}$	$A\{1,3\}$
B		$B\{1,2\}$	$B\{1,2\}$	$B\{1,2\}$
C			$C\{1,2,3\}$	$C\{1,3\}$
D				$D\{1,3\}$

Beginning with the full set $\{A, B, C, D, E\}$ is there anything that never wins any pairwise contest? A beats $\{C, D\}$, B beats $\{C, D, E\}$,

C beats $\{A, D, E\}$, D beats $\{A, E\}$ and E beats nothing. Thus obviously we have to remove E, and everything in $\{A, B, C, D\}$ beats at least one other option in that set.

Alternatively, we could start with one option that is in the top cycle and construct the sequence for beats. Let us begin with the one that is Pareto dominated, D. D beats $\{A, E\}$, E beats nothing thus can not be the second element of this chain, A beats $\{B, E\}$, B beats C and D, but we want the chain to include everything so C beats D and E, Thus the sequence is (D, A, B, C), notice this is the agenda except I tossed in E second—since anything beats it it does not matter where it goes.

From this point on you may assume that everyone votes as if they were always pivotal.

(e) *(7 points total)* Consider the standard committee model of selecting an option. In round one they vote to either accept or reject the first option in the agenda, with a majority vote determining which occurs. If they reject they do the same to the second option in the second round, and so on until if they reject the next to last option they are accepting the final option by default.

i. *(2 points)* In the fourth round, will they accept or reject the fourth option? Why?

Solution 10 (B, C) *from above we know B wins, thus B is accepted.*

ii. *(1 point)* In the third round, will they accept or reject the third option? Why?

Solution 11 *We know that if they reject A they will accept B, and from the table we know that a majority will vote for A over B. Thus A will be accepted.*

iii. *(1 point)* In the second round, will they accept or reject the second option?

Solution 12 *Rejecting E means accepting A, and a majority prefers A to E, thus E is rejected.*

iv. *(1 point)* In the first round, will they accept or reject the first option?

Solution 13 *Rejecting D means accepting A, and persons 2 and 3 prefer D, thus D is accepted.*

v. *(2 points)* What will be the option that is selected. Why?

Solution 14 *D will be accepted because it is the only strategy that survives backward induction over the stage games.*

(f) *(6 points total)* Consider the status quo model. In the first round they will vote between the first two options in the agenda, the option that gets the majority of the votes will become the *status quo*. Thereafter in the n'th round they will vote between the status quo and the $n+1$'st option.

i. *(2 points)* Considering the option that the status quo could be any of the options before it, what would be the outcome in a vote between that status quo and the final option in the agenda? For one of them explain why.
C versus A—C wins
C versus B—B wins
C versus D—C wins unanimously.
C versus E—C wins
Thus a vote for $\{A, D, E\}$ is actually a vote for C. A vote for B in the previous round will result in B being selected.

ii. *(2 points)* Given this analysis, now repeat the same exercise for the third round.
B versus A—this is actually B versus C because a vote for A will result in C being elected. Thus B wins.
B versus D—like above this is B versus C and B wins.
B versus E—like above this is B versus C and B wins.
Notice now that a vote for *anything* is actually a vote for B.

iii. *(1 point)* Do the same analysis for the second round.

Solution 15 *What they do does not matter, thus any voting pattern is part of a subgame perfect equilibrium.*

iv. *(1 point)* Finally, given all your previous analysis what will be the winner in the first round. Does it matter which wins?

Solution 16 *It can be anything, because it has not mattered for a while which one wins.*

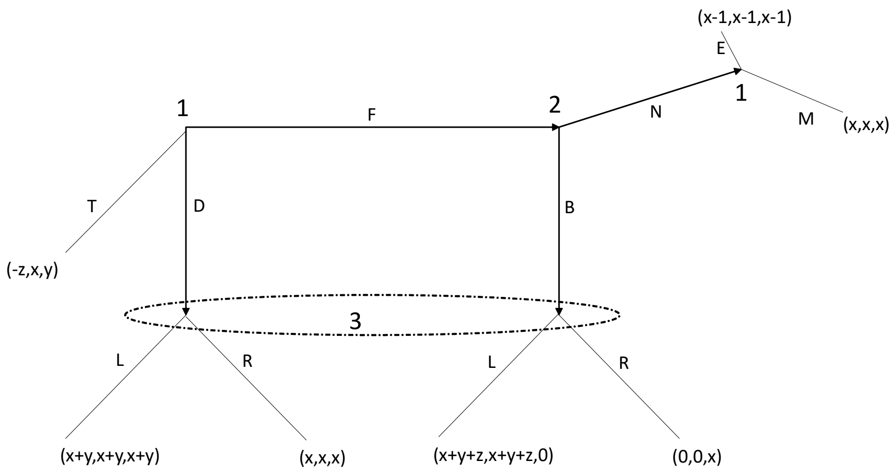
(g) *(2 points)* Now the people have decided to vote on whether to use the standard committee model or the status quo model. Which one would win and who would vote for which option?

Solution 17 *The committee model results in C, the status quo results in B, thus a majority will vote for the status quo because B is preferred to C.*

x	y	z	$\Pr_3(D\|D,B)$
1	2	3	$\left[0, \frac{1}{3}\right]$
2	3	4	$\left[0, \frac{2}{3}\right]$
3	1	4	$\left[0, \frac{3}{4}\right]$
4	4	1	$\left[0, \frac{1}{2}\right]$

4. *(22 points total)* Consider the general extensive form game below, In this game player 1 chooses first in the upper left hand corner, after that point

the choices flow in the direction of the arrows.



Remark 18 *Did you notice it looks like a horsey? It's Selten's horsey, the first game to explain why subgame perfection and Nash equilibrium were not enough in extensive form games. Indeed it is famous that the only equilibrium is to avoid player 3.*

(a) *(4 points)* Write down the strategies of this game.

Solution 19 $S_1 = \{T, D, F\} \times \{E, M\}$, $S_2 = \{B, N\}$, $S_3 = \{L, R\}$

(b) *(3 points)* If this was a game of perfect information (everyone knew what had happened at every decision point in the game) how would the strategies change?

Solution 20 *Now* $S_3 = \{L(D), R(D)\} \times \{L(B), R(B)\}$, *or we would need to specify player 3's response to* D *and* B.

(c) *(6 points)* Treating this as a game of perfect information, find the equilibrium strategies and write them out below.

Solution 21 *Since* $x + y > x$ $L(D)$, *and since* $x > 0$ $R(B)$. $s_3^* = (L(D), R(B))$
Furthermore, since $x > x - 1$ M *for player 1 at his final decision. Thus player 2 will choose* N, *and then player 1 will choose* D.
Thus $s^* = ((D, M), (N), (L(D), R(B)))$

(d) *(5 points)* Rewrite these strategies as strategies of the game as given. Are they a weak sequential equilibrium? Are they a subgame perfect equilibrium? Why or why not?

Solution 22 *The equilibrium path is* (D, L) *so we have to have* $s_3 = L$*, thus this strategy would be written as* $((D, M), (N), (L))$*. In order for it to be a weak sequential equilibrium player 2 must believe that player 3 will choose L if his information set is reached. In this case* N *is not optimal, thus he must switch to* B*. That is sufficient for it not to be a WSE, but continuing in this case P1 would switch to* F*, and then P3 would switch to R.*

It can be a subgame perfect equilibrium, or rather a Nash equilibrium since the game is a subgame. Player 2 will never take an action, so it is fine if he claims he will go N*. Or rather you could say that he believes* $\Pr(R|B) = 1$*, which again is fine because it is off the equilibrium path.*

(e) *(4 points)* Find the weak sequential equilibrium strategies, be sure to clearly specify the *range* of off-path beliefs that support this as an equilibrium.

Solution 23 *Since L did not work, let's try R. P1 will still choose* M*, so player 2 will choose* N*. At this point player 1 is indifferent between* F *and* D*, but if he chooses* D *then P3 will switch to L—and we don't want to go down that road. Thus the equilibrium is* $((F, M)(N)(R))$*. Finally what beliefs justify R? Let* $\Pr_3(D|D, B) = \beta$ *then:*

$$\begin{aligned} U_3(L, \beta) &= \beta U_3(D, L) + (1-\beta) U_3(F, B, L) \\ &= \beta(x+y) + (1-\beta) 0 = \beta(x+y) \\ U_3(R, \beta) &= \beta U_3(D, R) + (1-\beta) U_3(F, B, R) \\ &= \beta x + (1-\beta) x = x \end{aligned}$$

$$\begin{aligned} U_3(R, \beta) &\geq U_3(L, \beta) \\ x &\geq \beta(x+y) \\ \frac{x}{x+y} &\geq \beta \end{aligned}$$

Thus $0 \leq \Pr_3(D|D, B) \leq \frac{x}{x+y}$