ECON 439
Midterm: Normal Form Games

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This exam will start at about 10:15 and end 100 minutes later, around 11:55

Points will only be given for work shown.

1. *(2 points)* **Honor Statement:** Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test. I will also not offer assistance to others. Finally I will not use a calculator or other electronic aid for calculation during this test.

   Name and Surname: _
   Student ID: _
   Signature:

   _

2. *(9 points total)* About weakly dominated strategy.

   (a) *(6 points)* Explain the difference between a strategy being *dominated* and a strategy being *weakly dominated*.

   Solution 1 *Strategy a dominates strategy b if for all $s_{-i} \in S_{-i}$ $u_i(a, s_{-i}) > u_i(b, s_{-i})$. Strategy a weakly dominates strategy b if sometimes the strict inequality is replaced with equality.*

   (b) *(3 points)* Explain why it is not irrational for a person to use a weakly dominated strategy.

   Solution 2 *Given that you know what the other person is doing, playing a weakly dominated strategy gives you no loss, so why rule it out?*

   Especially since that sometimes means it might be a best response, the fundamental benchmark of rationality.

   And particularly because if you rule out the weakly dominated strategy you might not have a desirable Nash equilibrium—which might occur because the best response to the weakly dominant strategy might be different from the weakly dominated one.

   Most famously in the Bertrand game when price can be any real number the only Nash equilibrium is in weakly dominated strategies.

3. *(3 points)* What does it mean for an industry to be *production efficient?*

   Solution 3 *It means that the total output is produced at minimal cost. Any firms that produce output should have the same marginal cost.*

4. *(32 points total)* Consider the following Bayesian game. There are two states of the world, α and ω. Player 2 knows the state of the world, all player 1 knows is that $\Pr(\alpha) = \rho \in (0,1)$. I want you to solve this problem for all values of ρ. The game played in state α is:

α — Player 2

		a	b	c	z
Player 1	U	$-c; a$	$a; a+b+c$	$b+c; c$	$0; -a$
	D	$d; c$	$-a-b-c; -b$	$c; -a$	$a+b; b+c$

the game played in state ω is:

ω — Player 2

		a	b	c	z
Player 1	U	$a; a+c$	$a; -a$	$c; -b$	$0; a+b+c$
	D	$-b; -c$	$-a-b-c; b+c$	$b+c; b$	$a+b; -b$

α — Player 2

		a	b	c	z
Player 1	U	$-1; 3$	$3; 8$	$5; 1$	$0; -3$
	D	$1; 1$	$-8; -4$	$1; -3$	$7; 5$

ω — Player 2

		a	b	c	z
Player 1	U	$3; 4$	$3; -3$	$1; -4$	$0; 8$
	D	$-4; -1$	$-8; 5$	$5; 4$	$7; -$

$$\rho_U = \frac{7}{18}, \rho_D = \frac{11}{18}, p = \frac{9}{20}$$

α — Player 2

		a	b	c	z
Player 1	U	$0; -1$	$-3; 1$	$1; 6$	$5; 3$
	D	$3; 5$	$3; 3$	$-6; -2$	$3; -1$

ω — Player 2

		a	b	c	z
Player 1	U	$0; 6$	$1; 4$	$1; -1$	$3; -$
	D	$3; -2$	$-2; -3$	$-6; 5$	$5; 2$

$$\rho_D = \frac{3}{10}, \rho_U = \frac{7}{10}, p = \frac{1}{2}$$

α — Player 2

		a	b	c	z
Player 1	U	$2; 2$	$-7; -1$	$2; -4$	$5; 3$
	D	$-2; 4$	$4; 7$	$3; 2$	$0; -4$

ω — Player 2

		a	b	c	z
Player 1	U	$-1; -2$	$-7; 3$	$3; 1$	$5; -$
	D	$4; 6$	$4; -4$	$2; -1$	$0; 7$

$$\rho_U = \frac{5}{16}, \rho_D = \frac{11}{16}, p = \frac{4}{15}$$

α — Player 2

		a	b	c	z
Player 1	U	$5; 7$	$4; 4$	$-9; -3$	$4; -2$
	D	$0; -2$	$-4; 2$	$2; 9$	$7; 4$

ω — Player 2

		a	b	c	z
Player 1	U	$5; -3$	$-3; -4$	$-9; 7$	$7; 3$
	D	$0; 9$	$2; 6$	$2; -2$	$4; -$

$$\rho_D = \frac{5}{16}, \rho_U = \frac{11}{16}, p = \frac{10}{21}$$

(a) *(4 points)* Find all of the best responses of player 2, you may mark them on the tables above but you will loose two points if you do not explain your notation below.

Solution 4 *All the answers will be written for the abstract game:*

α		Player 2			
		a	b	c	z
Player 1	U	$-c; a$	$a; a+b+c^2$	$b+c; c$	$0; -a$
	D	$d; c$	$-a-b-c; -b$	$c; -a$	$a+b; b+c^2$

ω		Player 2			
		a	b	c	z
Player 1	U	$a; a+c$	$a; -a$	$c; -b$	$0; a+b+c^2$
	D	$-b; -c$	$-a-b-c; b+c^2$	$b+c; b$	$a+b; -b$

And I will write a 2 in the upper right hand corner when it is a best response for player 2.

Thus the two optimal strategies are $BR_2(U) = (b(\alpha), z(\omega))$, $BR_2(D) = (z(\alpha), b(\omega))$

In this problem a strategy for player 2 is written as $(x(\alpha), y(\omega))$ where $x \in \{a, b, c, z\}$ and $y \in \{a, b, c, z\}$

(b) *(10 points)* In the table below, across the top write player 2's key strategies, then in the appropriate box below write the expected utility of player 1 against these strategies. There are way more cells than necessary, and you can do calculations below the table. If x is one of these key strategies, then:

Strategies for P2→	$(b(\alpha), z(\omega))$	$(z(\alpha), b(\omega))$
$Eu(U, x)$	$\rho(a) + (1-\rho)(0) = a\rho$	$\rho(0) + (1-\rho)(a$
$Eu(D, x)$	$\rho(-a-b-c) + (1-\rho)(a+b) = a+b-2a\rho-2b\rho-c\rho$	$\rho(a+b) + (1-\rho$

(c) *(6 points)* Find the pure strategy Nash equilibria, and the range for ρ that they are equilibria.

Solution 5 $U \in BR_1(BR_2(U))$ *if:*

$$\begin{aligned} a\rho &\geq a+b-2a\rho-2b\rho-c\rho \\ a\rho-(-2a\rho-2b\rho-c\rho) &\geq a+b \\ 3a\rho+2b\rho+c\rho &\geq a+b \\ \rho &\geq \frac{a+b}{3a+2b+c} \end{aligned}$$

Likewise $D \in BR_1(BR_2(D))$ *if:*

$$2a\rho-b-c-a+2b\rho+c\rho \geq a-a\rho$$

$$\begin{aligned}
2a\rho + 2b\rho + c\rho - (-a\rho) &\geq a - (-b - c - a) \\
3a\rho + 2b\rho + c\rho &\geq 2a + b + c \\
\rho &\geq \frac{2a + b + c}{3a + 2b + c}
\end{aligned}$$

Let $\rho_U = \frac{a+b}{3a+2b+c}$ *and* $\rho_D = \frac{2a+b+c}{3a+2b+c}$ *then* $\rho_U < \rho_D$ *as long as* $a + c > 0$. *The complete answers are thus:*

$$NE = \begin{cases} (U, (b(\alpha), z(\omega))) & \text{if} \quad \rho \geq \frac{a+b}{3a+2b+c} \\ (D, (z(\alpha), b(\omega))) & \text{if} \quad \rho \geq \frac{2a+b+c}{3a+2b+c} \end{cases}$$

(d) *(4 points)* You should have found that for some range of ρ there is no pure strategy Nash equilibrium. Does this mean that there is no Nash equilibrium? Explain.

Solution 6 *There is always a Nash equilibrium, or at least in any game with a finite number of strategies. We have shown there is no pure strategy Nash equilibrium, thus the equilibrium must be mixed.*

(e) *(4 points)* Notice that in a mixed strategy Nash equilibrium in this game player 1 can only make player 2 indifferent between her choices in either state α or state ω. Which one should player 1 choose and why?

Solution 7 *The problem occurs when* $\rho = \Pr(\alpha)$ *is low, or when the state* ω *dominates. Looking at the payoffs of this game (after deleting unecessary strategies)*

ω		*Player 2*	
		b	z
Player 1	U	$a; -a^1$	$0; a + b + c^2$
	D	$-a - b - c; b + c^2$	$a + b; -b^1$

we can see a cycle in best responses or a mixed strategy equilibrium. If P1 plays the mixed strategy equilibrium here that means P2 is free to use any strategy that places positive probability on (b, z) *and we should be able to find an equilibrium.*

(f) *(4 points)* Let $p = \Pr_1(U)$, assuming player 2 only uses pure strategy best responses, find the optimal value for p in a mixed strategy equilibrium.

$$\begin{aligned}
U_2(p, b, \omega) &= p(-a) + (1 - p)(b + c) \\
U_2(p, z, \omega) &= p(a + b + c) + (1 - p)(-b)
\end{aligned}$$

$$p(-a) + (1 - p)(b + c) = p(a + b + c) + (1 - p)(-b)$$

$$p = \frac{2b + c}{2a + 3b + 2c}$$

α	β	μ	χ	τ	$p_1(p_2)$	$p_2(p_1)$	p_1^*	p_2^*
7	$\frac{1}{2}$	$\frac{1}{2}$	12	2	$\frac{1}{4}p_2 + 13$	$\frac{1}{4}p_1 + 8$	16	12
7	1	$\frac{2}{3}$	1	17	$\frac{1}{3}p_2 + 4$	$\frac{1}{3}p_1 + 12$	9	15
12	1	$\frac{2}{3}$	2	10	$\frac{1}{3}p_2 + 7$	$\frac{1}{3}p_1 + 11$	12	15
16	2	$\frac{1}{2}$	2	12	$\frac{1}{4}p_2 + 5$	$\frac{1}{4}p_1 + 10$	8	12

5. *(19 points)* In a given industry firms maximize their profits by choosing price, $p_i \geq 0$. Their demand curves are:

$$\begin{aligned} q_1 &= \alpha - \beta p_1 + \mu\beta p_2 \\ q_2 &= \alpha - \beta p_2 + \mu\beta p_1 \end{aligned}$$

and firm one has the cost function $c_1(q) = \chi q_1$ while firm 2 has the cost function $c_2(q) = \tau q_2$.

(a) *(4 points)* Set up the objective function of both firms. Notice that each firm maximizes over price.

$$\pi_1(p_1, p_2) = p_1 q_1 - \chi q_1 = (p_1 - \chi) q_1 = (p_1 - \chi)(\alpha - \beta p_1 + \mu\beta p_2)$$

$$\pi_2(p_1, p_2) = (p_2 - \tau)(\alpha - \beta p_2 + \mu\beta p_1)$$

(b) *(6 points)* Find the best response of each firm.

$$\begin{aligned} \frac{\partial \pi_1}{\partial p_1} &= (\alpha - \beta p_1 + \mu\beta p_2) - (p_1 - \chi)\beta \\ &= \alpha + \beta\chi - 2\beta p_1 + \beta\mu p_2 = 0 \end{aligned}$$

$$p_1 = \frac{1}{2\beta}(\alpha + \beta\chi + \beta\mu p_2)$$

$$\frac{\partial \pi_2(p_1, p_2)}{\partial p_2} = (\alpha - \beta p_2 + \mu\beta p_1) - \beta(p_2 - \tau) = 0$$

$$p_2 = \frac{1}{2\beta}(\alpha + \beta\tau + \beta\mu p_1)$$

(c) *(6 points)* Find the equilibrium prices.

$$\begin{aligned} p_1 &= \frac{1}{2\beta}\left(\alpha + \beta\chi + \beta\mu\left(\frac{1}{2\beta}(\alpha + \beta\tau + \beta\mu p_1)\right)\right) \\ p_1 &= \frac{1}{2}\chi + \frac{1}{4}\tau\mu + \frac{1}{2}\frac{\alpha}{\beta} + \frac{1}{4}\mu^2 p_1 + \frac{1}{4}\frac{\alpha}{\beta}\mu \end{aligned}$$

$$\begin{aligned} p_1 &= \frac{1}{4\beta - \beta\mu^2}(2\alpha + \alpha\mu + 2\beta\chi + \beta\tau\mu) \\ p_2 &= \frac{1}{2\beta}\left(\alpha + \beta\tau + \beta\mu\left(\frac{1}{4\beta - \beta\mu^2}(2\alpha + \alpha\mu + 2\beta\chi + \beta\tau\mu)\right)\right) \\ &= \frac{1}{\beta(4 - \mu^2)}(2\alpha + \alpha\mu + 2\beta\tau + \beta\mu\chi) \end{aligned}$$

(d) *(3 points)* Is this equilibrium Pareto efficient? Production efficient? Explain

Solution 8 *If both firms produce a positive output it is not production efficient because one of the firms has a strictly lower marginal cost. This means it is not Pareto efficient because the lower cost firm could always be contracted to produce all the output.*

6. *(24 points total)* Consider the following normal form game.

	α	β	γ	δ	ε
A	$8;5$	$7;6$	$4;1$	$-3;0$	$3;-5$
B	$4;8$	$-4;1$	$13;13$	$5;6$	$0;-4$
C	$3;6$	$3;8$	$14;-1$	$10;9$	$5;1$
D	$5;5$	$4;10$	$8;9$	$9;4$	$8;1$
E	$4;6$	$0;3$	$3;1$	$5;9$	$8;1$

	α	β	γ	$\delta\,(\alpha\beta\varepsilon)$	ε
A	$13;5^{1}$	$10;8^{12}$	$4;3$	$6;-5$	$-6;0$
B(E)	$4;8$	$0;6$	$6;3$	$11;3^{1}$	$5;14^{2}$
C	$4;11$	$-4;3$	$18;18^{2}$	$0;-4$	$7;8$
D	$6;8$	$6;11$	$21;-3^{1}$	$5;3$	$12;14^{12}$
E	$7;7$	$4;12^{2}$	$11;9$	$11;3^{1}$	$9;4$

	α	β	γ	δ	$\varepsilon\,(\alpha\beta\delta)$
A	$8;5^{1}$	$7;6^{12}$	$4;1$	$-3;0$	$3;-5$
B	$4;8$	$-4;1$	$13;13^{2}$	$5;6$	$0;-4$
C	$3;6$	$3;8$	$14;-1^{1}$	$10;9^{12}$	$5;1$
D	$5;5$	$4;10^{2}$	$8;9$	$9;4$	$8;1^{1}$
E(D)	$4;6$	$0;3$	$3;1$	$5;9^{2}$	$8;1^{1}$

	α	β	γ	δ	$\varepsilon\,(\alpha\beta\delta)$
A	$13;7^{1}$	$10;10^{12}$	$6;3$	$-4;0$	$4;-7$
B	$9;9$	$6;16^{2}$	$11;13$	$13;6$	$11;3^{1}$
C	$6;11$	$-6;3$	$20;20^{2}$	$9;10$	$0;-6$
D	$4;10$	$4;11$	$23;-3^{1}$	$16;14^{12}$	$7;3$
E(B)	$3;10$	$0;4$	$4;3$	$7;14^{2}$	$11;3^{1}$

	α	β	γ	$\delta\,(\alpha\beta\gamma)$	ε
A	$12;12^{1}$	$7;18^{2}$	$13;7$	$7;5^{1}$	$7;13$
B	$7;7$	$-7;5$	$12;11$	$0;-7$	$19;19^{2}$
C	$1;11$	$1;7$	$18;12^{12}$	$6;5$	$24;-5^{1}$
D	$13;6$	$8;11^{12}$	$-1;0$	$1;-6$	$7;5$
E(A)	$7;11$	$0;1$	$6;12^{2}$	$7;5^{1}$	$1;5$

(a) *(5 points)* Find each of the pure strategy best responses. You may mark them on the game above but you will loose two points if you do not explain your notation.

Solution 9 *I write a* 1 *(2 respectively) if it is a best response for player* 1 *(2 respectively).*

(b) *(2 points)* Find the unique dominated strategy in this game.

Solution 10 *See each game above, it is a strategy of player 2 (column player) and it is dominated by all but one of the other strategies.*

Remark 11 *They get zero points for this question if they choose the weakly dominated strategy for player 1.*

(c) *(2 points)* After removing the dominated strategy, find a unique strategy that is now dominated.

Solution 12 *This is a strategy for player 1 that was* weakly *dominated in the last stage. Since it was a best response against the dominated strategy of player 2 it is now dominated.*

(d) *(3 points)* Find the pure strategy Nash equilibria of this game. We write equilibria in strategies, not payoffs.

Solution 13 *These are the boxes with both a* 1 *and* 2 *in them. To be clear how they answer in the game:*

	α	β	γ	$\delta\,(\alpha\beta\varepsilon)$	ε
A	$13;5^1$	$10;8^{12}$	$4;3$	$6;-5$	$-6;0$
$B(E)$	$4;8$	$0;6$	$6;3$	$11;3^1$	$5;14^2$
C	$4;11$	$-4;3$	$18;18^2$	$0;-4$	$7;8$
D	$6;8$	$6;11$	$21;-3^1$	$5;3$	$12;14^{12}$
E	$7;7$	$4;12^2$	$11;9$	$11;3^1$	$9;4$

these are (A,β) *and* (D,ε)

7. *(8 points)* Prove that a Cournot equilibrium is production efficient if and only if it is symmetric. Just to be clear: in a Cournot equilibrium firms maximize their profits over quantity, and firm i's revenue is $R_i(q_i,Q) = P(Q)\,q_i$ where $q_i \geq 0$ is firm i's output, Q is the total output in the industry, and $P(Q)$ is the inverse demand curve with $P'(Q) < 0$.

Solution 14 *The profits of firm i are:*

$$\pi_i(q_i,Q) = P(Q)\,q_i - c_i(q_i)$$

and its first order condition is:

$$P + P'q_i - mc_i = 0$$

or:

$$mc_i = P + P'q_i$$

thus if $q_i = q_j = q$ *then*

$$mc_i = P + P'q = mc_j$$

and they have the same marginal costs. We also have:

$$q_i = \frac{1}{P'}(mc_i - P) = \frac{1}{|P'|}(P - mc_i)$$

so if $mc_i = mc_j = c$ *then:*

$$q_i = \frac{1}{P'}(c - P) = q_j$$

and the proof is done.

8. *(3 points for participation)* Standard feedback questions: Since the final exam will be cumulative I would like to know:

   Remark 15 *Since everyone did participate, I just added 3 points to question 2. You did get credit for your answers and I did read them all.*

   (a) What is the topic for the midterm that you understood the least?

   (b) What is the topic for the midterm you feel you understood the best?

   (c) Out of ten, how well do you feel the professor is explaining things?