ECON 439

Practice Questions, Normal Form Games

Dr. Kevin Hasker

These questions are supposed to help you prepare for exams and quizzes, they are not to be turned in. Answers will be posted before the relevant exam.

1 Chapter 2—Nash Equilibrium, Theory

1. Consider an arbitrary game $G = (I, A, u)$ where I is the finite set of players; A_i is the finite set of strategies for player i, $A = \times_{i\in I} A_i$; and $u_i : A \to R$ is i's payoff function, $u = [u_i]_{i\in I}$. Throughout you may assume players only use pure strategies.

   (a) Define the *rationality* of a strategy in this game.
   A strategy a_i is **rational** *if there are beliefs $\beta^i \in \times_{j\neq i} A_j$ such that $a_i \in \arg\max_{a\in A_i} u_i\left(a_i, \beta^i\right)$*

   (b) Define *correct expectations* in this game.
   Someone has correct expectations if $\beta^i = a_{-i}$, where a_{-i} are the actions people will actually take.

   (c) Define *best response* in this game.
   For given $\beta^i \in \times_{j\neq i} A_j$ $BR_i\left(\beta^i\right) = \arg\max_{a\in A_i} u_i\left(a_i, \beta^i\right)$

   (d) Write down two definitions of a (pure strategy) Nash equilibrium in this game. One that uses only best responses, and another one that does not use best responses. Explain why the two definitions are equivalent.
   a^* is a Nash equilibrium if for all i, $a_i^* \in BR_i\left(a_{-i}^*\right)$
   a^* is a Nash equilibrium if for all i, and all $a \in A_i$ $u_i\left(a_i^*, a_{-i}^*\right) \geq u_i\left(a, a_{-i}^*\right)$ **or**
   a^* is a Nash equilibrium if for all i, a_i^* is rational when β^i satisfies correct expectations.
   These definitions are formally equivalent because being rational means that you are best responding, and $\beta^i = a_{-i}^*$ summarizes correct expectations. Going from the other direction if $a_i^* \in BR_i\left(a_{-i}^*\right)$ then again we can let $\beta^i = a_{-i}^*$ and we satisfy rationality and correct expectations.

   (e) Explain the relationship between a Nash equilibrium and a social contract, being clear about the attributes of the social contracts you are discussing.
   A Nash equilibrium can be thought of as a self enforcing social contract. In other words it is the way you are expected to play the game, and we assume that everyone knows the contract. For example, when dating usually the boy has the pleasure/terror of being the one to

ask the girl out. The girl just has to sit there hoping she's dropped enough "subtle" hints that he knows what is expected of him. Of course... speaking from experience one never is really sure one knows the social contract a given girl is operating under. And I KNOW that girls in my past have been very frustrated by me not picking up on their signals... But that's reality, this is that radical simplification of reality called theory.

2. *(30 points total)* Consider the following game:

		α	$\beta\,(D)$	δ	γ
	A	$7;2^{1}$	$0;2$	$6;4^{2}$	$0;2$
1	B	$3;7^{2}$	$14;3^{1}$	$8;4^{1}$	$-1;4$
	$C\,(ID)$	$0;4$	$13;13$	$3;15^{2}$	$1;0$
	D	$3;2$	$6;1$	$6;3$	$\underline{2;4}^{12}$

(a) Find all the best responses of both parties, for **one** person and **one** strategy explain why it is a best response in detail. You may mark your answers on the table above, but explain your notation below.

I use a 1 in the upper right hand corner to indicate the best responses of player 1, a 2 to indicate the best responses of player 2. I will explain the best response of player 1 to α in game 1. A is the best response because $u_1(A,\alpha) > u_1(B,\alpha) = u_1(D,\alpha) > u_1(C,\alpha)$.

(b) Find the pure strategy Nash equilibrium of this game.

It is underlined in each of the games above. It is simply the square where both 1 and 2 are best responding.

(c) Find the unique dominated strategy in this game. Explain in detail why it is a dominated strategy, and explain carefully why no other strategies are dominated.

It is marked by a (D) in each game above. It is always the unique strategy for the column player that is never a best response to a pure strategy. For the others, if a strategy is a best response then it can not be dominated, so this only leaves one other option, the strategy that results in a Pareto Efficient symmetric payoff for the right choice of player 2. The only thing that is better against this critical strategy for player 2 is the best response for player 1, and this is a worse response to the other player playing the pure strategy Nash equilibrium, thus it is not dominated.

To go through these answers in game 1 in detail:

The dominated strategy is β, it is dominated because $u_2(A,\beta) < u_2(A,\delta)$; $u_2(B,\beta) < u_2(B,\delta)$; $u_2(C,\beta) < u_2(C,\delta)$; $u_2(D,\beta) < u_2(D,\delta)$. This is the only candidate because nothing else gives a higher payoff against C, since (C,β) is the symmetric Pareto Efficient payoff. The strategy C is not dominated because the only thing that gives a strictly higher payoff against δ is B, and $u_1(C,\gamma) > u_1(B,\gamma)$.

(d) Find the set of strategies that survives iterated deletion of dominated strategies.

In the new games we have only one action that is never a best response to pure strategies, it was the action for the row player mentioned in the last part of the question. Now that the column action that led to a symmetric Pareto efficient payoff has been eliminated, we can now consider the action in the pure strategy Nash equilibrium of this game. One can easily show that this Pareto dominates it, but I will not because I do not ask for that in this part of the question.

(e) *(4 points)* Find the unique mixed strategy Nash equilibrium of this game that does not have any weight on the strategies in the pure strategy Nash equilibrium.

For simplicity I will write down again the games after deleting (iterated) dominated strategies and the pure strategy Nash equilibrium:

		α	δ
1	A	$7;2^1 \rightarrow$	$6;4^2 \downarrow$
	B	$3;7^2 \uparrow$	$8;4^1 \leftarrow$

$p = \frac{1}{3}, q = \frac{3}{5}$

and in each of these games you can clearly see there is a cycle in the best responses, thus there is a mixed strategy Nash equilibrium (at least in potential, and in this game that will be a NE). The probabilities that the column player plays one of his actions is p*, the probability that the row player plays one of his actions is* q*, and these are written beside each game. I will go through the steps of solving for* p *and* q *for game 1.*

$$
\begin{aligned}
U_1(A,p) &= pu_1(A,\alpha) + (1-p)\,u_1(A,\delta) = p7 + (1-p)\,6 = p + 6 \\
U_1(B,p) &= pu_1(B,\alpha) + (1-p)\,u_1(B,\delta) = p3 + (1-p)\,8 = 8 - 5p
\end{aligned}
$$

$$
\begin{aligned}
p + 6 &= 8 - 5p \\
p &= \frac{1}{3}
\end{aligned}
$$

(f) A payoff is *Pareto Efficient* if there is no way to increase one person's payoff without hurting the other persons. Find the set of Pareto Efficient payoffs in this game.

This is very simple because the payoff of the dominated and iterated dominated strategy is very large. Thus it Pareto dominates all the payoffs except for the payoffs to a player either playing the dominated or the iterated dominated strategy. For example in game 1 *these are the strategy pairs:* $(C,\beta),(B,\beta),(C,\delta)$

(g) What does this game illustrate about the relationship between Nash equilibria and Pareto Efficiency? Explain why this tension exists.

It illustrates that Nash equilibrium is only Pareto efficient by coincidence. There may be a Nash equilibrium that is Pareto Efficient, but there may not be. This is because society can choose between the strategy pairs (C, β) and (D, γ) (for example). Player 1, on the other hand, can only choose between (C, β) and $(A, \beta), (B, \beta), (D, \beta)$ .Player 2 can then only make a choice given what player 1 chooses (in this case B) so no one in the game can make the choice and compare the costs and benefits like society can.

3. Consider the following Normal form game:

		Player 2		
		α	β	$\delta\,(\alpha, \beta)$
Player 1	A	$6;6^{2}$	$6;6^{12}$	$0;4$
	$B\,(\backslash\delta, A, D)$	$5;5$	$2;9^{2}$	$16;0^{1}$
	$C\,(B, D)$	$3;5^{2}$	$-3;4$	$0;3$
	D	$13;8^{12}$	$6;6^{1}$	$6;3$

(a) Find *all* the best responses for both players, you may mark them on the table above but explain your notation below.

I have marked a 1 in the upper right hand corner if it is a BR for 1, and a 2 in the upper right hand corner if it is a BR for 2. Notice that in this game there are multiple best responses to at least one of the opponent's strategies.

(b) For both players find a dominated strategy, and carefully explain why it is dominated.

In the table above I have marked this by a $X\,(Y, Z)$ X is what is dominated and it is dominated by both Y and Z. I will explain the logic carefully for the game:

		Player 2		
		α	β	$\delta\,(\alpha, \beta)$
Player 1	A	$6;6^{2}$	$6;6^{12}$	$0;4$
	$B\,(\backslash\delta, A, D)$	$5;5$	$2;9^{2}$	$16;0^{1}$
	$C\,(B, D)$	$3;5$	$-3;4$	$0;3$
	D	$13;8^{12}$	$6;6^{1}$	$6;3$

for all the other games it can be found by changing the names of the strategies and payoffs, but it will be essentially identical:

$\mu =$	α	β	δ
$u_2(\mu, A)$	6	6	4
$u_2(\mu, B)$	5	9	0
$u_2(\mu, C)$	5	4	3
$u_2(\mu, D)$	8	6	3

as you can see $u_2(\delta, X)$ for $X \in \{A, B, C, D\}$ *is always strictly below either other payoff, so both α and β dominate δ. Either answer, well explained, will give full credit.*

$X =$	A	B	C	D
$u_1(X,\alpha) =$	6	5	3	13
$u_1(X,\beta) =$	6	2	-3	6
$u_1(X,\delta) =$	0	16	0	6

As you can see $u_1(C,\mu)$ is always weakly lower than every other payoff, but it is only always strictly lower for B and D, thus B and D dominate C.

(c) After you remove those dominated strategies from the game, find one more dominated strategy and explain which action dominates it.

This strategy is only for player 1, and I have written $X(\backslash\mu, Y, Z)$ where μ is the action for Player 2 that was removed in the last round. The dominated action is always dominated by all the other actions in the remaining game. To go in more specific explanations for the game:

Player 1		Player 2: α	β
	A	$6;6^2$	$6;6^{12}$
	$B(\backslash\delta, A, D)$	$5;5$	$2;9^2$
	D	$13;8^{12}$	$6;6^1$

$X =$	A	B	D
$u_1(X,\alpha) =$	6	5	13
$u_1(X,\beta) =$	6	2	6

you can see that $u_1(B,\mu)$ is always strictly lower than either of the other payoffs for $\mu \in \{\alpha, \beta\}$.

(d) In this game explain why you should not remove strategies that are weakly dominated.

If you remove weakly dominated strategies in this game you will remove one of the Nash equilibria. While weak dominance is an interesting equilibrium refinement society could be stuck on playing a weakly dominated Nash equilibrium. Of course in this game the rational for playing the weakly dominated NE is weak (the NE is not "stable" for any reasonable definition of stability), but still it is possible.

4. Consider the following Normal form game:

Player 1		Player 2: α	β	ψ	δ
	A	$10;0$	$6;37$	$4;39$	$0;40$
	B	$8;3$	$4;6$	$6;7$	$37;4$
	C	$9;4$	$7;6$	$7;5$	$39;3$
	D	$0;10$	$5;9$	$3;8$	$40;0$

(a) Find the best responses for both parties, you may mark them on the table above. For at least one of them **carefully** justify why it is a best response below.

I will do this for the game

		Player 2			
		α	β	ψ	δ
	A	$10; 0^1$	$6; 37$	$4; 39$	$0; 40^2$
Player 1	B	$8; 3$	$4; 6$	$6; 7^2$	$37; 4$
	C	$9; 4$	$7; 6^{12}$	$7; 5^1$	$39; 3$
	D	$0; 10^2$	$5; 9$	$3; 8$	$40; 0^1$

and the best response to α. $u_1(A,\alpha) = 10 > u_1(C,\alpha) = 9 > u_1(B,\alpha) = 8 > u_1(D,\alpha) = 0$ *so* A *is the best response.*

(b) Are there any dominated strategies? If so list them along with what dominates them and **carefully** explain why.

Again, for the same game,

		Player 2			
		α	β	ψ	δ
	A	$10; 0^1$	$6; 37$	$4; 39$	$0; 40^2$
Player 1	B	$8; 3$	$4; 6$	$6; 7^2$	$37; 4$
	C	$9; 4$	$7; 6^{12}$	$7; 5^1$	$39; 3$
	D	$0; 10^2$	$5; 9$	$3; 8$	$40; 0^1$

The only strategy that is never a best response is B. *And you can see that* $u_1(C,\alpha) = 9 > u_1(B,\alpha) = 8$, $u_1(C,\beta) = 7 > u_1(B,\beta) = 4$, $u_1(C,\psi) = 7 > u_1(B,\psi) = 6$, $u_1(C,\delta) = 39 > u_1(B,\delta) = 37$

(c) Are there any strategies that can only be eliminated by iterated deletion of dominated strategies? If so list them along with what dominates them and **carefully** explain why.

There are actually no such strategies, once B *is removed the remaining game is:*

	α	β	ψ	δ
A	$10; 0^1$	$6; 37$	$4; 39$	$0; 40^2$
C	$9; 4$	$7; 6^{12}$	$7; 5^1$	$39; 3$
D	$0; 10^2$	$5; 9$	$3; 8$	$40; 0^1$

and the only candidate is ψ. *Now only* δ *is better against* A $u_2(A,\delta) = 40 > 39 = u_2(A,\psi)$, *but* δ *is not better against* D $u_2(D,\delta) = 0 < 8 = u_2(D,\psi)$

(d) Is there a Nash equilibrium in pure strategies? If so explain why it is a Nash equilibrium.

Yes, on every game there is one outcome that is a best response for both parties. This is a Nash equilibrium because it is at the intersection of best responses, no matter how many times and in what order you iterate the best responses you will stay at this point forevery.

(e) Is there a cycle in best responses? If so explain what it is.

In every game it is the cycle $(A,\alpha) \rightarrow_2 (A,\delta) \rightarrow_1 (D,\delta) \rightarrow_2 (D,\alpha) \rightarrow_1 (A,\alpha)$

(f) Find a mixed strategy Nash equilibrium of this game where actions that are not in a cycle over best responses have zero probability.

This value is p in the table above, and every game reduces to the following game when we only consider actions in the cycle.

	α	δ
A	$10; 0^1$	$0; 40^2$
D	$0; 10^2$	$40; 0^1$

let $\rho = \Pr(\alpha)$

$$\begin{aligned} U_1(A,p) &= \rho u_1(A,\alpha) + (1-\rho)\,u_1(A,\delta) = 10\rho \\ U_1(D,p) &= \rho u_1(D,\alpha) + (1-\rho)\,u_1(D,\delta) = 40(1-\rho) \\ U_1(A,p) &= U_1(D,p) \\ 10\rho &= 40(1-\rho) \\ \rho &= \frac{4}{5} \end{aligned}$$

let $\gamma = \Pr(A)$

$$\begin{aligned} U_2(\alpha,p) &= \gamma u_2(\alpha,A) + (1-\gamma)\,u_2(\alpha,D) = 10(1-\gamma) \\ U_2(\delta,p) &= \gamma u_2(\delta,D) + (1-\gamma)\,u_2(\delta,D) = 40\gamma \\ U_2(A,p) &= U_2(D,p) \\ 10(1-\gamma) &= 40\gamma \\ \gamma &= \frac{1}{5} \end{aligned}$$

5. Consider the following normal form game:

		Player 2: L	C	R
Player 1	U	$6; 4^1$	$11; 4^1$	$1; 5^2$
	M	$4; 4$	$8; 8^2$	$0; 6$
	D	$3; 5^2$	$6; 1$	$2; 4^1$

(a) Find the best responses of both players, **in at least one case explain why it is a best response**. You may mark your answers above but you will loose 2 points if you do not explain your notation below.

In the games I put a 1 in the upper right hand corner of player 1's BRs and a 2 in the upper right hand corner of player 2's BRs.

To explain in one case I should go through the payoffs of player 1 to the action L and explain that the one I chose is the highest.

(b) There is one dominated action in this game, find it and **carefully show that it is dominated**.

In every game there is only one action that is never a best response, so that is the only candidate for being dominated. I will delete the dominated action from each game. The dominating action is marked with a star.

Player 2

	L	C	R
U*	$6;4^1$	$11;4^1$	$1;5^2$
D	$3;5^2$	$6;1$	$2;4^1$

To carefully show why it is dominated I should go through the pair of payoffs for each action of player 2, and show that the dominating one always has a higher payoff.

(c) After elimination of that dominated action, there is one new action that is dominated, find it—you do not need to explain why it is dominated in great detail but you do need to tell me which action dominates it.

Again there is only one possible candidate for player 2, and the dominating action is marked with a star.

Player 2

	L	R*
U*	$6;4^1$	$1;5^2$
D	$3;5^2$	$2;4^1$

(d) Find the unique Nash equilibrium of this game.

The equilibrium is specified to the left of each game, but I will calculate it in each case.

$$p = \frac{1}{4}, q = \frac{1}{2}$$

Player 2

	L	R*
U*	$6;4^1$	$1;5^2$
D	$3;5^2$	$2;4^1$

$$p = \Pr(L), 6p + (1-p) = 3p + 2(1-p), p = \frac{1}{4}$$

$$q = \Pr(U), 4q + 5(1-q) = 5q + 4(1-q), q = \frac{1}{2}$$

6. Consider the following strategic form game:

		Player 2		
		α	β	δ
	A	$1;2$	$4;1$	$3;0$
Player 1	B	$3;3$	$3;2$	$5;4$
	C	$4;5$	$5;4$	$2;7$

(a) Find all the best responses to pure strategies, you may mark them above but explain your notation below.

		Player 2		
		α	β	δ
	A	$1;2^2$	$4;1$	$3;0$
Player 1	B	$3;3$	$3;2$	$5;4^{12}$
	C	$4;5^1$	$5;4^1$	$2;7^2$

The best responses for player 1 are marked with a 1 *above, for player 2 are marked with a* 2.

(b) Show that the unique Nash equilibrium is the only rational action to take in this game. *(Hint: dominated strategies.)*

A is never a best response for 1, and β is never a best response for 2. Thus these are the only actions that might be dominated strategies.

A—versus α $\{B, C\}$ are better, verus $\{\alpha, \beta\}$ only C is better, versus δ only B is better, so it is not dominated.

β—verusus A only α is better, and α is better that β for both B and C too, thus α dominates β.

	α	δ
A	$1;2^2$	$3;0$
B	$3;3$	$5;4^{12}$
C	$4;5^1$	$2;7^2$

Now the only action that is never a best response is A.

A—versus α $\{B, C\}$ are better, versus δ only B is better, so B dominates A.

	α	δ
B	$3;3$	$5;4^{12}$
C	$4;5^1$	$2;7^2$

α—now δ is always the best response in the remaining game, so δ dominates α.

	δ
B	$5;4^{12}$
C	$2;7^2$

B is the best response and thus the dominant strategy when the opponent has one action to consider.

	δ
B	$5;4^{12}$

since rational players will not play strategies that can be delted by iterated deletion of dominated strategies the only rational way to play this game is (B, δ).

Let me share another, perfectly fine, answer to this question.

		Player 2		
		α	β	δ
	A	$\mathring{1}; \hat{2}$	$4; \hat{1}$	$\mathring{3}; 0$
Player 1	B	$\mathring{3}; \underline{\hat{3}}$	$3; \hat{2}$	$\boxed{\mathring{5}}; \underline{4}$
	C	$4; \underline{\hat{5}}$	$5; \hat{4}$	$\boxed{2}; \underline{7}$

By comparing the hatted payoffs you can see that α dominates β. By comparing the payoffs with open dots over them you can see that B dominates A once β is removed. By looking at the best response or comparing the underlined payoffs it is clear that in the remaining game δ is a dominant strategy. By looking at the best response or comparing the boxed payoffs in the remaining game it is clear that B is a dominant strategy.

7. In the following Normal form game:

		Player 2				
		α	β	ψ	δ	ε
	A	$1;3$	$1;6$	$6;5$	$5;8$	$2;7$
Player 1	B	$3;2$	$5;5$	$9;0$	$5;1$	$9;4$
	C	$3;3$	$8;9$	$5;0$	$6;2$	$6;10$
	D	$4;3$	$7;1$	$6;0$	$4;2$	$8;2$
	E	$2;9$	$2;5$	$8;6$	$7;8$	$4;4$

(a) Find all of the best responses, you may mark them in the graph above.

				(1)	(2)	
		α	β	ψ	δ	ε
(1)	A	$1;3$	$1;6$	$6;5$	$5;8^2$	$2;7$
	B	$3;2$	$5;5^2$	$9;0^1$	$5;1$	$9;4^1$
	C	$3;3$	$8;9^1$	$5;0$	$6;2$	$6;10^2$
	D	$4;3^{12}$	$7;1$	$6;0$	$4;2$	$8;2$
(3)	E	$2;9^2$	$2;5$	$8;6$	$7;8^1$	$4;4$

The best responses for player 1 are marked with a 1 *above, for player 2 are marked with a* 2.

(b) Using iterated removal of dominated strategies, remove two strategies for each person. Explain your work and draw the new game on the next page in the table provided.

Player 2

Strategies of Player 2 $\rightarrow$

Player 1

Strategies of Player 1 $\uparrow$

__;__	__;__	__;__
__;__	__;__	__;__
__;__	__;__	__;__

I will go through my work in detail for the game:

	α	β	ψ	δ	ε
A	$1;3$	$1;6$	$6;5$	$5;8^2$	$2;7$
B	$3;2$	$5;5^2$	$9;0^1$	$5;1$	$9;4^1$
C	$3;3$	$8;9^1$	$5;0$	$6;2$	$6;10^2$
D	$4;3^{12}$	$7;1$	$6;0$	$4;2$	$8;2$
E	$2;9^2$	$2;5$	$8;6$	$7;8^1$	$4;4$

In the first round the only possible candidates are strategy A for player 1 and strategy ψ for player 2.
A—$\{B,C,D,E\}$ are better against α, $\{B,C,D,E\}$ are better against $\{\alpha,\beta\}$, $\{B,E\}$ are better against $\{\alpha,\beta,\psi\}$,$\{E\}$ is better against $\{\alpha,\beta,\psi,\delta\}$, and E dominates A because it is also better against ε.
ψ—$\{\beta,\delta,\varepsilon\}$ are better against $\{A\}$,$\{\beta,\delta,\varepsilon\}$ are better against $\{A,B\}$,$\{\beta,\delta,\varepsilon\}$ are better against $\{A,B,C\}$,$\{\beta,\delta,\varepsilon\}$ are better against $\{A,B,C,D\}$, $\delta$is better against $\{A,B,C,D,E\}$, so δ dominates ψ. The equivalent strategies in all games are marked by a (1) in the tables above.
Now lets look at the restricted game:

	α	β	δ	ε
B	$3;2$	$5;5^2$	$5;1$	$9;4^1$
C	$3;3$	$8;9^1$	$6;2$	$6;10^2$
D	$4;3^{12}$	$7;1$	$4;2$	$8;2$
E	$2;9^2$	$2;5$	$7;8^1$	$4;4$

The only strategy that is never a best response to any other pure strategy is δ.
δ—$\{\alpha,\beta,\varepsilon\}$ is better against $\{B\}$, $\{\alpha,\beta,\varepsilon\}$ is better against $\{B,C\}$,α is better against $\{B,C,D\}$,α is better against $\{B,C,D,E\}$, so α dominates δ. The equivalent strategies are marked by a (2) in the tables above.

	α	β	ε
B	$3;2$	$5;5^2$	$9;4^1$
C	$3;3$	$8;9^1$	$6;10^2$
D	$4;3^{12}$	$7;1$	$8;2$
E	$2;9^2$	$2;5$	$4;4$

E—$\{B,C,D\}$ are better against $\{\alpha\}$, $\{B,C,D\}$ are better against $\{\alpha,\beta\}$,$\{B,C,D\}$ are better against $\{\alpha,\beta,\varepsilon\}$, so now everything dominates E. These are the strategies marked with a (3) *above.*

The game is below for clarity.

	α	β	ε
B	$3;2$	$5;5^2 \downarrow$	$9;4^1 \longleftarrow$
C	$3;3$	$8;9^1 \longrightarrow$	$6;10^2 \uparrow$
D	$4;3^{12}$	$7;1$	$8;2$

(c) Find all of the pure strategy Nash equilibria.

There is only one, it is marked by both a 1 *and* 2 *in the upper right hand corner, i.e. (D,α).*

(d) Find a cycle in the best responses

This is marked by arrows in the table above.

(e) Find a *candidate* for a Nash equilibrium over the cycle you found in the last part of the question. (To be precise, only actions in the cycle have positive probability. By a candidate I mean that if their is a mixed strategy equilibrium over these actions then it must be this candidate.)

Again I will work on the game:

	α	β	ε
B	$3;2$	$5;5^2 \downarrow$	$9;4^1 \longleftarrow$
C	$3;3$	$8;9^1 \longrightarrow$	$6;10^2 \uparrow$
D	$4;3^{12}$	$7;1$	$8;2$

let $p = \sigma_2(\beta)$ then the payoffs of player 1 are:

$$\begin{aligned} U(B,p) &= 5p + 9(1-p) = 8p + 6(1-p) = U(C,p) \\ p &= \frac{1}{2} \end{aligned}$$

let $q = \sigma_1(B)$ then the payoffs of player 2 are:

$$\begin{aligned} U(\beta,q) &= 5q + 9(1-q) = 4q + 10(1-q) = U(\varepsilon,q) \\ q &= \frac{1}{2} \end{aligned}$$

and $p = q = \frac{1}{2}$ is the candidate of this game.

(f) Show that the candidate you found in the last part of this question is not actually a Nash equilibrium.

In order to be a Nash equilibrium the payoff you get from using the mixed strategy must be utility maximizing against your opponents mixed strategy. So we have to check this payoff versus the payoff of each players' other action.

$$U\left(\alpha, \frac{1}{2}\right) = 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right) = \frac{5}{2} < 7 = 5\left(\frac{1}{2}\right) + 9\left(\frac{1}{2}\right) = U\left(\beta, \frac{1}{2}\right)$$

but

$$U\left(D, \frac{1}{2}\right) = 7\left(\frac{1}{2}\right) + 8\left(\frac{1}{2}\right) = \frac{15}{2} > \frac{14}{2} = 8\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right) = U\left(C, \frac{1}{2}\right)$$

so this is not a Nash equilibrium. There is only one, pure strategy, Nash equilibrium in this game.

8. Consider the following model of firm location. There are two firms that choose a location at the same time: for $i \in \{a, b\}$, $l_i \in \{1, 2, 3, 4, 5, 6, 7\}$. Each firm's objective is to maximize its number of customers. Each consumer is endowed with a location ($v_k \in \{1, 2, 3, 4, 5, 6, 7\}$ $k \in (1, 2, 3, ..., , 30)$) and go to the firm that is closest to them, choosing each firm with equal likelihood if both firms are equally close. The number of consumers at each location is:

1	2	3	4	5	6	7
6	4	2	2	6	4	6

notice the total number of consumers is 30.

(a) Fill out the following table with the payoffs of firm a from being at location l when firm b is at location $m \in \{1, 2, 3, 4, 5, 6, 7\}$.

distribution	6	4	2	2	6	4	6
Location of firm b:	1	2	3	4	5*	6	7
Location of firm a:							
1	15	6	8	10	11	12	13
2	24	15	10	11	12	13	14
3	22	20	15	12	13	14	17
4	20	19	18	15	14	17	20
5	19	18	17	16	15	20	22
6	18	17	16	13	10	15	24
7	17	16	13	10	8	6	15

(b) Find the equilibrium by iterated removal of strictly dominated strategies.

Locations 2-4 strictly dominate 1. Removing 1 then locations 3-4 dominate 2. Removing 2 then 4 dominates 3. Removing 3 then 5 dominates 4.

Locations 2-6 dominate 7, removing 7 locations 2-5 dominate 6.

Thus 5 is the equilibrium.

9. In the following game find the best responses of both players, the Nash equilibrium (or equilibria) and any dominated strategies. You may mark the best responses, but list the Nash equilibrium (or equilibria) and dominated strategies below. For the dominated strategies you must also list the strategies that dominate them.

	α	β	ψ	δ
A	8,4	1,2	5,1	8,3
B	10,1	2,2	4,0	6,0
C	9,9	0,5	12,2	4,8
D	3,2	0,4	6,3	7,7

(a)

Player 2

		α	β	ψ	δ
	A	$8,4^2$	1,2	5,1	$8,3^1$
Player 1	B	$10,1^1$	$2,2^{1,2}$	4,0	6,0
	C	$9,9^2$	0,5	$12,2^1$	4,8
	D	3,2	0,4	6,3	$7,7^2$

Best responses have been stated above.

Player 1 has no dominated strategies but for Player 2, ψ is dominated by β, Since all the payoffs of Player 2 in the β column is larger than ψ column.

The only Nash Equilibrium of this game is (B, β) where the payoff is (2,2), because it is the only case where the best responses coincides

10. Consider the following strategic form game:

Player 2

		α	β	χ	δ
	A	10,8....	0,10....	3,8......	2,8......
Player 1	B	1,2......	2,6......	4,7......	5,7......
	C	3,0......	1,1......	0,0......	5,0......
	D	2,0......	4,4......	3,8......	6,3......
	E	2,2......	4,4......	3,5......	7,5......

(a) Find the best responses of both players, you may mark them in the game above or write them down below.

Player 2

		α	β	χ	δ
	A	$10,8^1$....	0,10....	3,8......	2,8......
Player 1	B	1,2......	2,6......	$4,7^{1,2}$......	$5,7^2$......
	C	3,0......	$1,1^2$......	0,0......	5,0......
	D	2,0......	$4,4^1$......	$3,8^2$......	6,3......
	E	2,2......	$4,4^1$......	$3,5^2$......	$7,5^{1,2}$......

1. (a) Find the Nash equilibria (they are all in pure strategies.)

   The Nash Equilibria of this game are (B,χ) and (E,δ) where the best responses coincides for both players.

   (b) Write down the definition of a (strictly) dominated strategy.

   In a strategic form game a strategy a_i of Player i is strictly dominated if there exists another strategy a_i' for Player i which gives better payoff whatever the other players chooses. In formal, $u_i(a_i, a_{-i}) < u_i(a_i', a_{-i})$ for every a_{-i} which is a list of strategies of all players except i

   (c) Remove all strictly dominated strategies and iterate the process until there are no strategies that are strictly dominated in the remaining game. You may mark out the strategies in the game above but indicate the order you remove them and what strategy dominates them below.

   We can observe that α is dominated by β for Player 2 since 8<10, 2<6, 0<1, 0<4 and 2<4, so let us eliminate α. Now similarly A is dominated by B and C is dominated by D for Player 1 so we can eliminate A and C too. The table we have now is the following.

Player 2

		β	χ	δ
Player 1	B	2,6......	4,7......	5,7......
	D	4,4......	3,8......	6,3......
	E	4,4......	3,5......	7,5.....

   And finally we can claim that β is dominated by χ since 6<7, 4<8, 4<5. So the final table will be as below.

Player 2

		χ	δ
Player 1	B	4,7......	5,7......
	D	3,8......	6,3......
	E	3,5......	7,5......

   We have eliminated the dominated strategies by following order and causes($\alpha < \beta$, A<B,C<D,$\beta < \chi$)

   (d) Write down the definition of a weakly dominated strategy.

   Weakly dominated strategies is similar to strictly dominated strategies, but weakly dominated strategies also allow the equality of payoffs for some of the cases. Indeed in a strategic form game a strategy a_i of Player i is weakly dominated if there exists another strategy a_i' for Player i which never gives worse payoff whatever the other players chooses. In formal, $u_i(a_i, a_{-i}) \leq u_i(a_i', a_{-i})$ for every a_{-i} which is a list of strategies of all players except i. There must be at least one a_{-i} where $u_i(a_i, a_{-i}) < u_i(a_i', a_{-i})$

(e) Write down two games that can be derived by removing weakly dominated strategies.

		Player 2: χ	δ
Player 1	B	$4,7^{12}$	$5,7^{2}$
	$E \geq D$	$3,5^{2}$	$7,5^{12}$

We can not remove any further strategies because now χ and δ give the same payoffs to player 2. The notation $E \geq D$ means that E weakly dominates D. It gives 3 against χ and 7 versus 6 against δ.

		Player 2: $\chi \geq \delta$
Player 1	B	$4,7^{12}$
	D	$3,8^{2}$
	E	$3,5^{2}$

.

		Player 2: $\chi \geq \delta$
Player 1	$B > E, D$	$4,7^{12}$

(f) Why is the concept of iterated removal of strictly dominated strategies an implication of rationality? Why is the concept of weakly dominated strategies not an implication of rationality?

In the example above you found that *which* weakly dominated action you removed first affected the final set of weakly undominated strategies. This means you must know both:

i. That other players are rational

ii. The order in which they are going to remove weakly dominated actions.

The order of removal matters a lot because the Nash equilibrium changes when you change the order of removal. In the first example above the NE were (B, χ) and (E, δ). In the second one the NE was (B, χ). Is (E, δ) a potential equilibrium or not?

In contrast one of the most important results for strictly dominated actions is that no matter which order you remove them in you always are left with the same set of strictly undominated actions. Thus all I need to know is that my opponent is rational.

2 Chapter 3—Nash Equilibrium, Illustrations.

1. Consider the following game of joint production. Two workers are considering how much to contribute to a project, of which each of them is entitled half the benefits. Worker i contributes $w_i \geq 0$ to the project, and then if $W = w_1 + w_2$ the benefit of the project is $R(W) = W(44 - 2W)$, and worker i receives $R(W)/2$. The costs of worker 1 are $c_1(w_1, w_2) = 1w_1^2$, the costs of worker 2 are $c_2(w_1, w_2) = 5w_2^2$.

(a) Set up the objective functions of both workers.

$$\max_{w_1} \frac{1}{2}(w_1+w_2)(44-2(w_1+w_2)) - 1w_1^2$$
$$\max_{w_2} \frac{1}{2}(w_1+w_2)(44-2(w_1+w_2)) - 5w_2^2$$

(b) Find their best responses.

$$\begin{aligned}
\frac{1}{2}(44-2(w_1+w_2)) - 2\frac{1}{2}(w_1+w_2) - 21w_1 &= 0 \qquad (1)\\
\frac{1}{2}44 - 2w_1 - 2w_2 - 2w_1 &= 0\\
22 - 2w_2 &= 4w_1\\
\frac{22}{4} - \frac{2}{4}w_2 &= w_1
\end{aligned}$$

:

$$\begin{aligned}
\frac{1}{2}(44-2(w_1+w_2)) - (w_1+w_2) - 2*5w_2 &= 0 \qquad (2)\\
22 - 2w_1 - 2w_2 - 10w_2 &= 0\\
22 - 2w_1 &= (10+2)w_2\\
\frac{22}{12} - \frac{2}{12}w_1 &= w_2
\end{aligned}$$

:

(c) Find the Nash equilibrium outputs of both workers.

$$\begin{aligned}
\frac{1}{2}\frac{44}{(4)} - \frac{2}{(4)}\left(\frac{1}{2}\frac{44}{(12)} - \frac{3}{(12)}w_1^*\right) &= w_1^*\\
\frac{4}{(4)(12)}w_1^* + \frac{44}{(4)(12)}5 &= w_1^*\\
\frac{11}{12}5 &= w_1 - \frac{1}{12}w_1^*\\
\frac{11}{12}5 &= \frac{11}{12}w_1^*\\
5 &= w_1^*
\end{aligned}$$

:

$$\begin{aligned}
\frac{1}{2}\frac{44}{(12)} - \frac{2}{(12)}w_1^* &= w_2^*\\
\frac{11}{6} - \frac{1}{(6)}(5) &= w_2^*\\
1 &= w_2^*
\end{aligned}$$

(d) *(4 points)* An outcome is *production efficient* if given the level of output it is not possible to reduce the costs of producing this output. Is the Nash equilibrium production efficient? Why or why not?

We want to minimize costs given $w_1+w_2 = W$, *and this is equivalent to:*

$$\min_{w_1} w_1^2 + 5\left(W - w_1\right)^2$$

$$2w_1 = 10\left(W - w_1\right) = 10w_2$$

plugging in the equilibrium values:

$$\begin{aligned} 2w_1^* &= 2\left(5\right) \\ 10w_2^* &= 10\left(1\right) = 10 \end{aligned}$$

(e) An outcome is *Pareto efficient* if it maximizes the sum of the profit of the two workers. Is the Nash equilibrium Pareto efficient? Why or why not?

For this question we need to maximize the sum of payoffs,

$$\left(w_1 + w_2\right)\left(44 - 2\left(w_1 + w_2\right)\right) - w_1^2 - 5w_2^2$$

the first order conditions are:

$$\left(44 - 2\left(w_1 + w_2\right)\right) - 2\left(w_1 + w_2\right) - 2w_1 = 0 \qquad (3)$$

$$\left(44 - 2\left(w_1 + w_2\right)\right) - 2\left(w_1 + w_2\right) - 10w_2 = 0 \qquad (4)$$

and at this point we don't have to go any further because we can see that equation 1 is always lower than 3 and likewise for 2. This means that worker 1 (2) will always work too little. However I know that those of you who solve this probably won't stop there. You'll probably solve it in its full glory. One thing we can see immediately by subtracting these equations is:

$$2w_1 = 10w_2$$

or $w_2 = \frac{1}{5}w_1$, *plugging this in:*

$$\begin{aligned} \left(44 - 2\left(w_1 + \frac{1}{5}w_1\right)\right) - 2\left(w_1 + \frac{1}{5}w_1\right) - 2w_1 &= 0 \\ 44 - 4w_1 - 2w_1 - 4\frac{1}{5}w_1 &= 0 \\ \frac{44}{4+2+4\frac{1}{5}} &= w_1^w \\ 44\frac{1}{4+10+10} &= w_2^w \end{aligned}$$

which are not the same as above

.

2. Consider the classic Bertrand game. Market demand is given by $Q = 64 - 4P$ and there are two firms with the same cost function, $c_i(q_1, q_2) = 2q_i$. Firms compete by choosing price $p_1 \geq 0$ and $p_2 \geq 0$ and for any (p_1, p_2):

$$q_i(p_1, p_2) = \begin{cases} 64 - 4p_i & \text{if} \quad p_i < p_j \\ 32 - 2p_i & \text{if} \quad p_i = p_j \\ 0 & \text{if} \quad p_i > p_j \end{cases}$$

(a) In a general game, define a *weakly dominated strategy.*

A strategy weakly dominates another if it always gives at least as high a payoff and sometimes gives a strictly higher payoff.

(b) In this game find the Nash equilibrium. (Hint: it is a weakly dominated strategy in this game.)

i. *In this problem each firm maximizes:*

$$\max_{p_i} (p_i - c)\, q_i(p_1, p_2)$$

and the Nash equilibrium is $p_1 = p_2 = 2$, *one can show that this is a Nash equilibrium fairly easily, If* $p_2 = 2$ *then as long as* $p_1 \geq 2$ $\Pi_1 = 0$, *thus* $p_1 = 2$ *is optimal. Or you can find this by noticing that if* $p_i > 2$ *then it weakly dominates* $p_i = 2$. *This is because if* $p_j > p_i > 2$ *you will get strictly positive profits, unlike when* $p_i = 2$. *Notice that actually any* $p_i > 2$ *weakly dominates any* $p_i \leq 2$, *but we can't have an equilibrium with* $p_i = p_j < 2$*because then the firms will make negative profits.*

(c) Now firm 2 advertises that they will always match the price of firm 1. Find the Nash equilibrium of this game given this. (Hint: You don't need to optimize for firm 2 at all.)

Due to firm 2's advertisement firm one can just maximize profits assuming that $p_2 = p_1$:

$$\max_{p_1} (p_1 - 2)(32 - 2p_1)$$

$$\begin{aligned} (32 - 2p_1) - 2(p_1 - 2) &= 0 \\ 9 &= p_1^* \end{aligned}$$

this is the monopoly price in this market.

(d) Comment on how this illustrates the difference between out of equilibrium logic and equilibrium logic.

In an out of equilibrium since, it's obviously best for me to go to a store with a price matching guarantee. Thus out of equilibrium this guarantee is always good. However in equilibrium this decreases the amount of competition for other stores, and thus results in a higher price in the market. Thus in equilibrium this is bad, out of equilibrium it is good.

3. (Asymmetric Competition) Consider a market where there are two firms producing substitutes. The primary difference in the firms is the strategic variable they use to maximize their profit. Firm b is a Bertrand competitor, and maximizes her profit over price. Firm c is a Cournot competitor, and maximizes his profit over quantity. Their cost functions are $c_1(q_1, q_2) = \chi q_1$, $c_2(q_1, q_2) = \chi q_2$, or they have the same constant marginal cost. The demand curve for firm B is:

$$q_B = 15 - \frac{1}{2}p_B - \frac{1}{2}q_C \ ,$$

notice I write it in terms of q_C because that is the strategic variable of firm C. The inverse demand curve for firm C is:

$$p_C = 15 + \frac{1}{2}p_B - \frac{3}{2}q_C \ .$$

(a) Show that the demand curves are symmetric. I.e. that they are based on a symmetric underlying model of the relationship between prices and quantities.

From the demand for C:

$$q_c = 10 + \frac{1}{3}p_B - \frac{2}{3}p_C$$

Plugging this into the demand for firm B:

$$\begin{aligned} q_B &= 15 - \frac{1}{2}p_B - \frac{1}{2}\left(10 + \frac{1}{3}p_B - \frac{2}{3}p_C\right) \\ &= 10 - \frac{2}{3}p_B + \frac{1}{3}p_C \end{aligned}$$

and these demand curves are clearly the same.

(b) Given that the demand curves are symmetric, is this a symmetric problem? Why or why not?

It is not a symmetric problem because the firms are not maximizing over the same strategic variable, to see this look at the objective functions below. You can clearly see: $\frac{\partial^2 \Pi_B}{\partial q_C \partial p_B} = -\frac{1}{2} < 0$ *while* $\frac{\partial^2 \Pi_C}{\partial q_C \partial p_B} = \frac{1}{2} > 0$, *thus the two problems are as different as different could be.*

(c) Set up the objective function of both firms.

$$\begin{aligned} \pi_i &= (p_i - 4)\, q_i \\ \pi_C &= \left(15 + \frac{1}{2}p_B - \frac{3}{2}q_C - 4\right) q_C \\ \pi_B &= (p_B - 4)\left(15 - \frac{1}{2}p_B - \frac{1}{2}q_C\right) \end{aligned}$$

(d) Find the best responses for both firms.

$$\begin{aligned}
\frac{\partial \pi_C}{\partial q_C} &= \left(15 + \frac{1}{2}p_B - \frac{3}{2}q_C - 4\right) - \frac{3}{2}q_C = 0 \\
3q_c &= 15 - 4 + \frac{1}{2}p_B \\
q_c &= 5 - \frac{1}{3}4 + \frac{1}{6}p_B
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \pi_C}{\partial p_B} &= \left(15 - \frac{1}{2}p_B - \frac{1}{2}q_C\right) - \frac{1}{2}(p_B - 4) = 0 \\
p_B &= 15 + 2 - \frac{1}{2}q_C
\end{aligned}$$

(e) Find the equilibrium prices and quantities.

$$\begin{aligned}
q_c &= 5 - \frac{4}{3} + \frac{1}{6}\left(15 + 2 - \frac{1}{2}q_C\right) \\
q_c &= \frac{30}{4} - 1 - \frac{1}{12}q_C \\
\left(1 + \frac{1}{12}\right) q_c &= \frac{30}{4} - 1 \\
q_c &= \frac{90}{13} - \frac{12}{13} \\
p_b &= 15 + 2 - \frac{1}{2}\left(\frac{90}{13} - \frac{12}{13}\right) \\
&= \frac{150}{13} + \frac{32}{13} = 14
\end{aligned}$$

$$\begin{aligned}
p_c &= 15 + \frac{1}{2}p_B - \frac{3}{2}q_C \\
&= 15 + \frac{1}{2}\left(\frac{150}{13} + \frac{32}{13}\right) - \frac{3}{2}\left(\frac{90}{13} - \frac{12}{13}\right) \\
&= \frac{270}{26} + \frac{68}{26} = 13
\end{aligned}$$

$$\begin{aligned}
q_B &= 15 - \frac{1}{2}\left(\frac{90}{13} - \frac{12}{13}\right) - \frac{1}{2}\left(\frac{150}{13} + \frac{32}{13}\right) \\
&= \frac{150}{26} - \frac{20}{26} = 5
\end{aligned}$$

(f) Find the profits of both firms. Which firm makes the lower profits?

$$\begin{aligned}
\pi_B &= (p_B - 4)\, q_B = \left(\frac{150}{13} + \frac{32}{13} - 4\right)\left(\frac{150}{26} - \frac{20}{26}\right) = \frac{25}{338}(30-4)^2 = 13 * 25 \\
\pi_C &= (p_C - 4)\, q_C = \left(\frac{270}{26} + \frac{68}{26} - 4\right)\left(\frac{90}{13} - \frac{12}{13}\right) = \frac{27}{338}(30-4)^2 = 13 * 27
\end{aligned}$$

and since $25 < 27$ *and so if you advise the bertrand competitor to switch to quantity competition you would be right. However without knowing the equilibrium in the alternative this advice is wrong.*

In this section I will calculate the answers for several wrong methodologies, this should never show up in any answer key I am merely doing it for grading purposes.

$$\begin{aligned} q_c &= \frac{5}{2} + \frac{1}{4} p_b \\ p_b &= 4q_c - 70 \end{aligned}$$

$$p_b = 4\left(\frac{5}{2} + \frac{1}{4} p_b\right) - 70$$

, No solution found.

$$\begin{aligned} \max\left(10 - \frac{2}{3} p_B + \frac{1}{3} p_C\right)(p_b - 4) &= \\ \left(10 - \frac{2}{3} p_b + \frac{1}{3} p_c\right) - \frac{2}{3}(p_b - 4) &= 0 \\ \frac{30}{4} + 2 + \frac{1}{4} p_c &= p_b \end{aligned}$$

$$\max\left(30 - q_B - 2q_C - 4\right) q_c$$

$$\begin{aligned} (30 - q_b - 2q_c - 4) + (-2) q_c &= 0 \\ \frac{30}{4} - 1 - \frac{1}{4} q_b &= q_c \end{aligned}$$

$$\begin{aligned} p_b &= \frac{30}{4} + 2 + \frac{1}{4} p_c \\ q_c &= \frac{30}{4} - 1 - \frac{1}{4} q_b \\ p_c &= 15 + \frac{1}{2} p_b - \frac{3}{2} q_c \\ q_b &= 15 - \frac{1}{2} p_b - \frac{1}{2} q_c \end{aligned}$$

$$\begin{aligned} p_b &= \frac{30}{4} + 2 + \frac{1}{4} p_c \text{ used} \\ q_c &= \frac{30}{4} - 2 - \frac{1}{4} q_b \\ p_c &= \frac{150}{7} + \frac{8}{7} - \frac{12}{7} q_c \\ q_b &= \frac{90}{8} - 1 - \frac{1}{8} p_c - \frac{1}{2} q_c \end{aligned}$$

$$
\begin{aligned}
p_b &= \frac{30}{4} + 2 + \frac{1}{4}p_c \text{ used} \\
q_c &= \frac{30}{4} - 1 - \frac{1}{4}q_b \text{ used} \\
p_c &= \frac{60}{7} + \frac{20}{7} + \frac{3}{7}q_b \\
q_b &= \frac{60}{7} - \frac{4}{7} - \frac{1}{7}p_c
\end{aligned}
$$

$$
\begin{aligned}
p_b &= \frac{270}{26} + \frac{68}{26} \\
q_c &= \frac{150}{26} - \frac{20}{26} \\
p_c &= \frac{150}{13} + \frac{32}{13} \\
q_b &= \frac{90}{13} - \frac{12}{13}
\end{aligned}
$$

$$
\begin{aligned}
116 - 2q_c &= p_b \\
\frac{5}{3} + \frac{1}{6}p_b &= q_c \\
116 - 2\left(\frac{5}{3} + \frac{1}{6}p_b\right) &= p_b \\
\frac{169}{2} &= p_b \\
\frac{5}{3} + \frac{1}{6}\left(\frac{169}{2}\right) &= q_c \\
\frac{63}{4} &= q_c
\end{aligned}
$$

(f) Consider the following Hotelling linear city. Customers are distributed over locations the $(1,2,3,4,5)$, and always buy from the firm that is closest to their location, if two **firms** are an equal distance from them they buy from each firm with equal probability. The distribution of customers is

Locations	1	2	3	4	5
Number of Customers	10	12	14	2	4

firms choose a location (or locations) to maximize the number of customers it gets, call their payoff π.

1. The Standard Model: In this model two firms, a and b, both choose a location simultaneously l_a is a's location and l_b is b's location.

   (a) Fill out the table below, in the box I want you to write in the payoff to firm 1 if firm 2 is in the location across the top of the table and firm

1 is in the location on the side. Notice I ONLY want you to write in the payoff of firm 1, if you are not clear on what you should do please ask. **Notice that there is a lot of symmetry in this table, if you can figure it out it will make answering it much easier.**

if $l_b =$	1	2	3	4	5
and $l_a = 1$ then $\pi_a =$	21	10	16	22	29
and $l_a = 2$ then $\pi_a =$	32	21	22	29	36
and $l_a = 3$ then $\pi_a =$	26	20	21	36	37
and $l_a = 4$ then $\pi_a =$	20	13	6	21	38
and $l_a = 5$ then $\pi_a =$	13	6	5	4	21

(b) Find the Nash equilibrium by iterated deletion of dominated strategies. **Be careful to make your answers to this question clear. If you mark your last table so much that I can not clearly see what you have written I will mark you down**.
2 always gives a higher payoff than 1 so 1 is dominated. By symmetry it must also be dominated for player 2. 4 also dominates 5, and now that 5 and 1 are removed 3 dominates 4. Now that 1,4, and 5 are removed 2 dominates 3, thus $l_a = l_b = 2$ must be the Nash equilibrium.

2. A Big Firm/Small Firm Model: In this model firm a only has one location, l_a, but firm b can choose two locations, l_{b1} and l_{b2}—however (for simplicity) we require that $l_{b1} = l_a$. Note that $l_{b1} = l_{b2} = l_a$ is allowed.

To be clear about how the division of subjects is done, assume that $l_{b1} = l_a = 5$ and $l_{b2} = 3$, then firm b would get all the consumers in locations 1-3 (36) half the consumers in location 5 (2) because l_{b1} and l_a are equally far from them, and half the consumers in location 4 (1) because **firms** *a and b are equidistant from them. Notice that firm b will have two branches and firm a will only have one branch equally far from the consumers in location 4, but still each firm gets half the customers.*

(a) For each location of firm a find the best responses of firm b. Explain the logic behind your best responses below the table.

if $l_a =$	1	2	3	4	5
The BR of firm b is $l_{b1} =$	1	2	3	4	5
$l_{b2} =$	2	3	2	3	4

First of all if $l_{b2} = l_{b1}$ this will cause no increase in the number of customers, thus $l_{b2} \neq l_{b1}$. Now it is clear that to capture the most customers we want $l_{b2} \in \{l_a - 1, l_a + 1\}$ and we want l_{b2} to be on the side where there are the most customers. You could also answer this question by finding the entire payoff matrix, but this explanation is simpler.

(b) For each best response you found above find the optimal location for firm a. Notice that a does not have to choose to be at one of the two locations b has chosen. Explain your logic below the table.

Just to be absolutely clear, let me give an example. Say that $l_{b1} = 1$ *and* $l_{b2} = 4$*, then firm a is free to choose* $l_a = 3$*, and when calculating the payoffs we do not assume that* $l_{b1} = 3$*. In this case* $\pi_a = 20$*,* $\pi_b = 22$.

if $l_{b1} =$	1	2	3	4	5
and $l_{b2} =$	2	3	2	3	4
$BR_a(l_{b1}, l_{b2})$	3	2	2	2	3

There are two types of choices that firm a could want to make. Either $l_a \in \{l_{b1}, l_{b2}\}$ *or* $l_a \in \{\min(l_{b1}, l_{b2}) - 1, \max(l_{b1}, l_{b2}) + 1\}$*. First of all, let* $D(l_{bi})$ *be the number of customers that go to branch* $i \in \{1, 2\}$ *given the current locations. If* $D(l_{b1}) > D(l_{b2})$ *and* $l_a \in \{l_{b1}, l_{b2}\}$ *then* $l_a = l_{b1}$*. If* $l_a \in \{\min(l_{b1}, l_{b2}) - 1, \max(l_{b1}, l_{b2}) + 1\}$ *then we want to put it on the side where there is the most customers. Which of the two strategies is optimal has to be decided based on the case.*

(c) Find the pure strategy Nash equilibria of this game.
If $l_{b1} = l_a = 2$ *and* $l_{b2} = 3$ *then this is the Nash equilibrium, looking at the best responses of b to the best responses of a you can clearly see that this is the one situation where the result is also the original strategy of a in the first table.*

(d) Do you think there will always be a pure strategy Nash equilibrium in this model? Why or why not? You can argue either answer, points will be given for the coherence of your argument.
Let me say that I know the answer to be that there will not always be a pure strategy Nash equilibrium in this model. Depending on the distribution of customers there may be a mixed strategy Nash equilibrium. You will have to trust me that this is not only because we require $l_{b1} = l_a$.
But why? The basic problem is that in equilibrium we must have the fact that it is better for firm a to match with one of the locations of firm b. It is possible that they will never want to do this, and then there can not be a pure strategy Nash equilibrium.
Just to prove my point, consider the distribution of customers:

Locations	1	2	3	4	5
Number of Customers	10	12	4	2	14

this is derived from the original one by switching the customers at location 3 to 5 and vice-versa. Then let us go through the two steps again:

if $l_a =$	1	2	3	4	5
The BR of firm b is $l_{b1} =$	1	2	3	4	5
$l_{b2} =$	2	3	2	3	4

if $l_{b1} =$	1	2	3	4	5
and $l_{b2} =$	2	3	2	3	4
$BR_a(l_{b1}, l_{b2})$	3	4	4	2	3

Now you can see there is no intersection of best responses, basically because now the incentive to not be in the same location as one of the b branches dominates. Now we will only have an equilibrium in mixed strategies. $(l_a, l_{b1}, l_{b2}) = (2,2,3) \to_b (4,2,3) \to_a (4,4,3) \to_b (2,4,3) \to_a (2,2,3)$. *Heck, I can even calculate it:*

$$\begin{aligned}
p &= \Pr(2,3), 1-p = \Pr(3,4) \\
U_a(2,p) &= p11 + (1-p)22 = p16 + (1-p)8 = U_a(4,p) \\
p &= \frac{14}{19} \\
q &= \Pr(2), 1-q = \Pr(4) \\
u_b((2,3), q) &= q31 + (1-q)26 = q20 + (1-q)34 = u_b((3,4), q) \\
q &= \frac{8}{19}
\end{aligned}$$

Wasn't that fun?

In a second price auction there is one item that will be given to one of I bidders. The bidder $i \in \{1,2,3,4,...,I\}$ has a value for the good $v_i \in [\underline{v}, \overline{v}]$ and submits a bid $b_i \in [\underline{v}, \overline{v}]$ .For simplicity you should assume the bid can only be in kurus, but that $\overline{v} - \underline{v}$ is very large relative to one kurus. Everyone involved in the auction knows everyone else's value, and you can assume without loss of generality that $v_1 > v_2 > v_3 ... > v_I$. A bidder's payoffs are $v_i - \max_{j \neq i} b_j$ if he wins (he pays the second highest bid), $\frac{1}{K}(v_i - \max_{j \neq i} b_j)$ if he ties for the highest bid with K other people, and 0 otherwise. Notice that a bidder's own bid never actually determines the price he pays.

1. Prove that bidding $b_i = v_i$ is a *weakly* dominant strategy.

   We will show that $b_i = v_i$ is always a best response, let $b_{(2)} = \max_{j \neq i} b_j$. If $b_{(2)} > v_i$ then person i does not want to win and pay $b_{(2)}$, and since $b_i = v_i < b_{(2)}$ they will not. Thus it is a best response. If $b_{(2)} = v_i$ then person i does not care whether he wins or not, and since $b_i = v_i = b_{(2)}$ it is a best response. If $b_{(2)} < v_i$ then person i wants to win, and $b_i = v_i > b_{(2)}$ guarantees that they do.

2. Prove that if for all i, $b_i = v_i$, then this is a Nash equilibrium.

   Solution method 1: Since above I showed that $b_i = v_i$ was always a best response if everyone uses this strategy everyone is obviously best responding, so this must be a Nash equilibrium.

   Solution method 2: In this equilibrium person 1 will win and receive the net payoff $v_1 - v_2 > 0$. Therefore person 1 will not want to lower their bid below the bid of person 2 because then they would win zero. Persons 2 and above will not want to increase their bid enough to win because then they would win $v_j - v_1 < 0$ for $j \in \{2,3,4,...,I\}$

3. Prove that if for any $j \in \{1,2,3,4,...,I\}$ $b_j = \overline{v}$ and for everyone else, $k \in \{1,2,3,4,...,I\} \backslash j$, $b_k = \underline{v}$ then this is a Nash equilibrium.

Person j will win the auction and get $v_j - \underline{v} > 0$, thus they will not want to lower their bid to $\underline{v}$ and risk losing the auction. If person $k \in \{1, 2, 3, 4, ..., I\} \backslash j$ raises their bid to $\overline{v}$ then they will receive $\frac{1}{2}(v_k - \overline{v}) < 0$ so they will not want to do that. Thus this is an equilibrium.

There are two firms that are Bertrand competitors in a market where market demand is $Q = 56 - 4p$. If they match price, they split demand, otherwise the firm that has the lower price gets all of the demand. Price must be in integers (for $i \in \{1, 2\}$, $p_i \in (0, 1, 2, 3, ...)$) and the constant marginal cost of production is 6. Just to be clear, the firm level demand is:

$$d_1(p_1, p_2) = \begin{cases} 0 & p_1 > p_2 \\ \frac{1}{2}(56 - 4p_1) & p_1 = p_2 \\ 56 - 4p_1 & p_1 < p_2 \end{cases}, d_2(p_1, p_2) = \begin{cases} 56 - 4p_2 & p_1 > p_2 \\ \frac{1}{2}(56 - 4p_2) & p_1 = p_2 \\ 0 & p_1 < p_2 \end{cases}$$

1. Set up the objective function of firm one, you can use the general demand curve $d_1(p_1, p_2)$.

$$\pi_1(p_1, p_2) = d_1(p_1, p_2)(p_1 - 6)$$

2. Find the monopoly price, i.e. the price a firm would charge if the other firm did not set a price.

$$\max_{p_1} (56 - 4p_1)(p_1 - 6)$$

$$\begin{aligned} (56 - 4p_1) - 4(p_1 - 6) &= 0 \\ 56 + 24 &= 8p_1 \\ 3 + 7 &= p^M = 10 \end{aligned}$$

3. For an arbitrary $p_2 \in (1, 2, 3, ...)$ find the profits for firm one from charging the price $p_2 - 1$, p_2, and $p_2 + 1$. Your answers should all have p_2 in them. (You should find three functions, one for $p_2 + 1$, one for p_2, and one for $p_2 - 1$).

$$\begin{aligned} \pi_1(p_2 + 1, p_2) &= 0 \\ \pi_1(p_2, p_2) &= \frac{1}{2}(56 - 4p_2)(p_2 - 6) \\ \pi_1(p_2 - 1, p_2) &= (56 - 4(p_2 - 1))(p_2 - 1 - 6) \end{aligned}$$

4. Find the best response of firm one for every $p_2 \in (0, 1, 2, 3, ...)$. Provide a precise mathematical proof when p_2 is equal to

   (a) the monopoly price plus one,

$$\begin{aligned} \pi_1(p_2 + 1, p_2) &= 0 \\ \pi_1(p_2, p_2) &= \frac{1}{8(4)}\left(56^2 - 2 * 56 * 4 * 6 + 4^2 6^2 - 4 * 6^2\right) = \frac{55}{2} \\ \pi_1(p_2 - 1, p_2) &= \frac{1}{4 * 4}(56 - 4 * 6)^2 = 64 \\ 0 &< \pi_1(p_2, p_2) < \pi_1(p_2 - 1, p_2) \end{aligned}$$

By definition the monopoly price must be higher, as can also be verified in each of your exams.

(b) the monopoly price,

$$\begin{aligned}
\pi_1(p_2+1,p_2) &= 0\\
\pi_1(p_2,p_2) &= \frac{1}{32}(56-4*6)^2 = 32\\
\pi_1(p_2-1,p_2) &= \frac{1}{4*4}\left(56^2-2*56*4*6+4^2 6^2-4*4^2\right) = 60\\
0 &< \pi_1(p_2,p_2) < \pi_1(p_2-1,p_2)
\end{aligned}$$

This is true, and is obvious for your exam, but I will need to verify it more carefully in the abstract case.

$$\begin{aligned}
\frac{1}{8*4}(56-4*6)^2 &< \frac{1}{4*4}\left(56^2-2*56*4*6+4^2 6^2-4*4^2\right)\\
(56-4*6)^2 &< 2\left(56^2-2*56*4*6+4^2 6^2-4*4^2\right)\\
0 &< 2\left(56^2-2*56*4*6+4^2 6^2-4*4^2\right)-(56-4*6)^2\\
0 &< 56^2-2*56*4*6+4^2 6^2-8*4^2 = 896
\end{aligned}$$

This is always true as long as $56 > 2\sqrt{2}*4+4*6$, *which it is for all of your exams.*

(c) marginal cost plus two,

$$\begin{aligned}
\pi_1(p_2+1,p_2) &= 0\\
\pi_1(p_2,p_2) &= 56-2*4-4*6 = 24\\
\pi_1(p_2-1,p_2) &= 56-4-4*6 = 28
\end{aligned}$$

:*and here it is obvious that* $\pi_1(p_2-1,p_2) > \pi_1(p_2,p_2) > 0$, *notice that* $56 > 2\sqrt{2}*4+4*6 > 4+4*6 = 28$

(d) marginal cost plus one,

$$\begin{aligned}
\pi_1(p_2+1,p_2) &= 0\\
\pi_1(p_2,p_2) &= \frac{1}{2}(56-4-4*6) = 14\\
\pi_1(p_2-1,p_2) &= 0
\end{aligned}$$

here it is equally obvious that $p_1 = p_2$ *is best.*

(e) marginal cost,

$$\begin{aligned}
\pi_1(p_2+1,p_2) &= 0\\
\pi_1(p_2,p_2) &= 0\\
\pi_1(p_2-1,p_2) &= 4*6-4-56 = -36
\end{aligned}$$

notice (importantly) that here any $p_1 \geq p_2$ *gives the same profit, zero.*

(f) marginal cost minus one.

$$\begin{aligned}\pi_1(p_2+1,p_2) &= 0\\ \pi_1(p_2,p_2) &= \frac{1}{2}*4*6-\frac{1}{2}*4-\frac{1}{2}*56=-18\\ \pi_1(p_2-1,p_2) &= 2*4*6-4*4-2*56=-80\end{aligned}$$

and here $p_1 = p_2 + t$ *for* $t \geq 1$ *is the optimal price.*

For the rest of the prices you can just generalize the results you found for these prices.

$$BR_1(p_2) = \begin{cases} p^m & \text{if} \quad p_2 > p^m \\ p_2 - 1 & \text{if} \quad p^m \geq p_2 > 6+1 \\ p_2 & \text{if} \quad p_2 = 6+1 \\ [6,\infty) & \text{if} \quad p_2 = 6 \\ [p_2+1,\infty) & \text{if} \quad p_2 < 6 \end{cases}$$

5. Find the Nash equilibria of this game.

   By intersecting the best responses the only strategies that survive are $p_1 = p_2 \in \{6,7\}$

6. Prove that setting price equal to the constant marginal cost is a weakly dominated strategy. (A strategy is weakly dominated if there is another strategy that always does at least as well and sometimes does strictly better.)

   If $p_1 = 6$ *then profits are* $\pi_1(p_1,p_2) = d_1(6,p_2)(6-6) = 0$ *independent of* p_2*. On the other hand choose any* $p_1 > c$*. Then if* $p_2 < p_1$ $\pi_1 = 0$*, if* $p_2 \geq p_1$ *profits are at least* $\frac{1}{2}(56-4p_1)(p_1-6) > 0$*. Thus* $p_1 = 6$*is weakly dominated by any* $p_1 > 6$*.*

α	χ	$p(2)$	$\Pr(0 \text{ cont.}\mid 2)$	$p(3)$	$\Pr(0 \text{ cont.}\mid 3)$	$p(I)$
12	3	$\frac{3}{4}$	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{1}{8}$	$1-\left(\frac{1}{4}\right)^{\frac{1}{I-1}}$
18	2	$\frac{8}{9}$	$\frac{1}{81}$	$\frac{2}{3}$	$\frac{1}{27}$	$1-\left(\frac{1}{9}\right)^{\frac{1}{I-1}}$
9	1	$\frac{8}{9}$	$\frac{1}{81}$	$\frac{2}{3}$	$\frac{1}{27}$	$1-\left(\frac{1}{9}\right)^{\frac{1}{I-1}}$
9	4	$\frac{5}{9}$	$\frac{16}{81}$	$\frac{1}{3}$	$\frac{8}{27}$	$1-\left(\frac{4}{9}\right)^{\frac{1}{I-1}}$

Consider the following war of attrition. Two people are fighting over an object which has a value of 1 to the winner. They can fight 0,1,2,3,4,5, or 6 periods, and the cost of a period of fighting is $\frac{3}{4}$ for person 1 and $\frac{2}{7}$ for person 2. Their strategy is the number of periods they will fight, $t_i \in \{0,1,2,3,4,5,6\}$ where $i \in \{1,2\}$. To be clear their utility function is:

$$u_1(t_1,t_2) = \begin{cases} 1-\frac{3}{4}t_2 & \text{if} \quad t_1 > t_2 \\ \frac{1}{2}\left(1-\frac{3}{4}t_2\right) & \text{if} \quad t_1 = t_2 \\ -\frac{3}{4}t_1 & \text{if} \quad t_1 < t_2 \end{cases}, u_2(t_1,t_2) = \begin{cases} 1-\frac{2}{7}t_1 & \text{if} \quad t_1 < t_2 \\ \frac{1}{2}\left(1-\frac{2}{7}t_1\right) & \text{if} \quad t_1 = t_2 \\ -\frac{2}{7}t_2 & \text{if} \quad t_1 > t_2 \end{cases}$$

1. Find the best responses for both persons by filling in the table below.
   I will answer this in general.Let $\chi_1 = \frac{3}{4}, \chi_2 = \frac{2}{7}$*. If* $t_{-i} > \frac{1}{\chi_i}$ *then the only best response is* $t_i = 0$*. If* $t_{-i} < \frac{1}{\chi_i}$ *then any* $t_i > t_{-i}$ *is optimal.*
2. Find all of the equilibria. You may either characterize them all or list them one by one.
   In the equilibria either $t_1 = 0, t_2 > \frac{1}{\chi_1}$ *or* $t_2 = 0$ $t_1 > \frac{1}{\chi_2}$

Consider a public good game where each person in the society can contribute $d_i \geq 0$ (to be precise $d_i \in [0, \infty)$), let $D = \Sigma_{i=1}^{I} d_i$ be the total amount contributed. The benefit to each person in the society of D is $B(D) = 80D - 2D^2$, a person gets this benefit no matter how much she or he contributes. For person i, the cost of contributing is $c_i(d_i) = 48d_i$. The objective of each person is to maximize their net benefit, which is defined as the benefit function minus the cost of contributing.

1. First assume that there are two people in the society ($I = 2$).
   (a) For person one, set up her objective function and find the first order condition.

$$\max_{d_1} 80D - 2D^2 - 48d_1 = \max_{d_1} 80(d_1 + d_2) - 2(d_1 + d_2)^2 - 48d_1$$

$$80 - 4(d_1 + d_2) - 48 = 0$$

   (b) Find the best response for one of the people.

$$\begin{aligned} 80 - 4(d_1 + d_2) - 48 &= 0 \\ d_1 &= \frac{1}{2}\frac{80 - 48}{2} - d_2 \\ d_1 &= 8 - d_2 \end{aligned}$$

   (c) Find the set of dominated strategies in the game. Explain your reasoning.
   Notice that d_1 *is decreasing in* d_2*, so when* d_2 *is at its lowest this is the highest amount* 1 *will ever want to contribute, this is:*

$$\begin{aligned} d_1 &= 8 - 0 \\ &= 8 \end{aligned}$$

   so any amount above this amount is dominated by contributing this amount or less.
   (d) Find the amount each person contributes in equilibrium, and the total amount contributed in equilibrium.

I will use symmetry:

$$\begin{aligned} d &= 8 - d \\ d &= 4 \\ D &= 2d = 8 \end{aligned}$$

however if you don't use symmetry you realize that for both parties the only restriction on equilibrium is:

$$d_1 + d_2 = 8$$

So the complete answer (not necessary for full credit is

$$d_1 \in [0, 8], d_2 = 1 - d_1$$

but notice that in any case $D = 4$ does not change.

2. Now assume that there are $I \geq 2$ people in this society.

   (a) For person one, set up her objective function and find the first order condition. You may denote the total amount contributed by the rest of the people as D_{-1}.

$$\max_{d_1} 80D - 2D^2 - 48d_1 = \max_{d_1} 80(d_1 + D_{-1}) - 2(d_1 + D_{-1})^2 - 48d_1$$

$$80 - 4(d_1 + D_{-1}) - 48 = 0$$

   (b) Find the best response for one of the people.

$$\begin{aligned} 80 - 4(d_1 + D_{-1}) - 48 &= 0 \\ d_1 &= 8 - D_{-1} \end{aligned}$$

   (c) Why can we assume that the amount contributed by all people will be the same in equilibrium?
   because of symmetry (notice that there is a symmetric equilibrium, there are others, but one does exist.)

   (d) Find the amount each person contributes in equilibrium, and the total amount contributed in equilibrium. Your answer should be a function of I, the total number of people in the society. *Be sure to check that if $I = 2$ your answer is the same in parts a and b.*
   by symmetry $D_{-1} = (I - 1)d$

$$\begin{aligned} d &= 8 - ((I - 1)d) \\ d &= \frac{1}{4I}(32) \\ D &= Id = 8 \end{aligned}$$

more generally the set of equilibria are $D = 8$ $d_i \geq 0$ for all i.

3. The social welfare function in this society is the sum of the net benefits of individuals, or the sum of their individual objective functions. I now want you to find the socially optimal amount to contribute for all $I \geq 2$.

   (a) Set up the objective function and show that it can be written only in terms of D, the total amount contributed.

$$\Sigma_{i=1}^{I}\left(80D - 2D^2 - 48d_i\right) = 80ID - 2ID^2 - 48\Sigma_{i=1}^{I} d_i = 80ID - 2ID^2 - 48D$$

$$\max_D \left(80ID - 2ID^2 - 48D\right)$$

   (b) Find the first order condition and the socially optimal total amount to contribute.

$$\begin{aligned} 80I - 4ID - 48 &= 0 \\ D &= 20 - 12\frac{1}{I} \end{aligned}$$

   (c) For each I compare the amount you found in part c to the amount you found in part b. Which is higher? Why is this?

$$\begin{aligned} D_c &= 20 - 12\frac{1}{I} > 20 - 12 = D_b\backslash \\ 12\frac{1}{I} &< 12 \\ \frac{1}{I} &< 1 \\ 1 &< I \end{aligned}$$

   The reason is this is that society cares about the fact that all I people get the same benefit from the contributions, while each individual only cares about their benefit. Notice that the marginal cost of contributing is the same. Also notice that in the socially optimal equilibrium no one suffers. To see this consider the equal split equilibrium, where $d_i = \frac{D}{I}$

$$\begin{aligned} 80D - 2D^2 - 48\frac{D}{I} &= \frac{D}{I}(80I - 48 - 2ID) = \frac{D}{I}\left(80I - 48 - 2I\left(20 - 12\frac{1}{I}\right)\right) \\ &= \frac{D}{I}\frac{1}{2}(80I - 48) \end{aligned}$$

   which is always positive when $80 > 48$*, But people are donating too much because the total donations are too high.*

Consider a Cournot Oligopoly where the inverse demand curve is given by $P = 17 - Q$ and the costs of a type a firm is $c_a\left(q^a, q^b\right) = q^a$ and the costs of a type b firm is $c_b\left(q^a, q^b\right) = 3q^b$.

1. In this part of the question assume that there is one firm of both types.

   (a) Set up the objective function of both firms.

$$\max_{q^a} \left(17 - \left(q^a + q^b\right)\right) q^a - q^a \; ; \; \max_{q^b} \left(17 - \left(q^b + q^a\right)\right) q^b - 3q^b$$

   (b) Find the best responses of both firms.

$$\begin{aligned} \left(17 - \left(q^a + q^b\right)\right) - q^a - 1 &= 0 \\ 2q^a &= 17 - 1 - q^b \\ q^a &= 8 - \frac{1}{2}q^b \end{aligned}$$

$$\begin{aligned} \left(17 - \left(q^b + q^a\right)\right) - q^b - 3 &= 0 \\ 2q^b &= 17 - 3 - q^a \\ q^b &= 7 - \frac{1}{2}q^a \end{aligned}$$

   (c) Find the Nash equilibrium quantities.

$$\begin{aligned} q^a &= 8 - \frac{1}{2}\left(7 - \frac{1}{2}q^a\right) \\ \frac{3}{4}q^a &= \frac{1}{4}(18) \\ q^a &= 6 \end{aligned}$$

$$\begin{aligned} q^b &= 7 - \frac{1}{2}(6) \\ &= 4 \end{aligned}$$

   (d) Find the profit of both firms in the Nash equilibrium.

$$\begin{aligned} Q &= q^a + q^b = 6 + 4 \\ &= 10 \\ P &= 17 - Q = 17 - (10) \\ &= 7 \\ \pi^a &= (P - 1)\, q^a \\ &= 36 \\ \pi^b &= (P - 3)\, q^b \\ &= 16 \end{aligned}$$

2. Now assume that there are two firms of type b, firm 1 and firm 2, firm 1 produces q_1^b and firm 2 produces q_2^b.

   (a) Set up the objective function of both types of firms.

$$\max_{q^a} \left(17 - \left(q^a + q_1^b + q_2^b\right)\right) q^a - q^a \ ; \ \max_{q_1^b} \left(17 - \left(q_1^b + q_2^b + q^a\right)\right) q_1^b - 3q_1^b$$

   (b) Find the best responses of both types of firms.

$$\begin{aligned} \left(17 - \left(q^a + q_1^b + q_2^b\right)\right) - q^a - 1 &= 0 \\ 2q^a &= 17 - 1 - \left(q_1^b + q_2^b\right) \\ q^a &= 8 - \frac{1}{2}\left(q_1^b + q_2^b\right) \end{aligned}$$

$$\begin{aligned} \left(17 - \left(q^a + q_1^b + q_2^b\right)\right) - q_1^b - 3 &= 0 \\ 2q_1^b &= 17 - 3 - \left(q^a + q_2^b\right) \\ q_1^b &= 7 - \frac{1}{2}\left(q^a + q_2^b\right) \end{aligned}$$

   (c) Why can you assume that $q_1^b = q_2^b$ in equilibrium?
   *Because of **symmetry**, they have the same objective function up to replacing q_1^b with q_2^b and vice-a-versa.*

   (d) Using the insight in the last part of the question, find the Nash equilibrium quantities.

$$\begin{aligned} q^a &= 8 - \frac{1}{2}\left(2q^b\right) \\ q^b &= 7 - \frac{1}{2}\left(q^a + q^b\right) \\ q^b &= 7 - \frac{1}{2}\left(8 - q^b + q^b\right) \\ &= 3 \\ q^a &= 8 - 3 \\ &= 5 \end{aligned}$$

   (e) Find the profit of both types of firms in the Nash equilibrium.

$$\begin{aligned} Q &= 5 + 3 + 3 \\ &= 11 \\ P &= 17 - (11) \\ &= 6 \end{aligned}$$

$$\begin{aligned}
\pi^a &= (P-1)\,q^a \\
&= 5*5=25 \\
\pi^b &= (P-3)\,q_b \\
&= 3*3=9
\end{aligned}$$

3. Now assume that the costs of firms of type b is $c_b\left(q^a,q^b\right)=3q^b+F$, where F is a fixed cost. Further consider a *free entry equilibrium* where as many firms of type b can enter as want. (Note that only one firm of type a will be in the market, and that the costs of that type of firm do not change.) This means that for firms of type b, $\pi_i^b=0$ in equilibrium.

   (a) If $F=9$, what will be the equilibrium number of firms of type b? Why?
   As $\pi^b=(P-3)\,q_b-9=0$.for 2 firms, there will be two firms

   (b) If $F=20$, what will be the equilibrium number of firms of type b? What will be total quantity produced by firms in the market?
   In every version $20>\pi^b$ *so no type b firms will enter, leaving the type a firm as a monopolist:*

$$\max_{q^a}\,(17-q^a)\,q^a-q^a$$

$$\begin{aligned}
17-2q^a-1 &= 0 \\
q^a &= 8
\end{aligned}$$

Consider a second price auction. The auctioneer has one item to give to one of I bidders, the bidders each submit a simultaneous bid of $b_i\in[0,100]$ and the item is rewarded to the highest bidder at the second highest bidder's bid. If more than one bidder is tied for the highest bid the item is awarded to one of them at random. Note that if i is the highest bidder then $\max_{j\neq i}b_j$ is the second highest bid. Each bidder has a value $v_i\in(0,100)$ and their utility is:

$$u_i\left(b\right)=\begin{cases}0 & \text{if they are not the highest bidder}\\ v_i-\max_{j\neq i}b_j & \text{if they are the unique highest bidder}\\ \frac{1}{J}\left(v_i-\max_{j\neq i}b_j\right) & \text{if there are } J \text{ people who tie for the highest bid.}\end{cases}$$

Assume that $v_1>v_2>v_3>...>v_I$ and that $I>5$. Remember that in this game everyone knows everyone else's v_i.

1. Find a *symmetric* equilibrium strategy and prove that it is an equilibrium.
   This symmetric equilibrium is $b_i=v_i$. *First of all, the high bidder has no incentive to change their bid because it does not affect the price he pays. If he lowers it too much he looses the auction at a price he is willing to pay. So this is a best response. Second of all, all other bidders will have to raise their bid to* $b_j>v_1$ *to win the auction but this will result in a negative utility. Thus their bid is also a best response.*

2. Find an asymmetric equilibrium where bidder 5 wins the auction and prove that it is an equilibrium.

   If $b_5 = 100$ and $b_j = 0$ for $j \neq 5$ then bidder 5 will have no incentive to change his bid since his bid does not affect the price he pays, since $v_5 > 0$ he does not want to lower his bid until he doesn't win. All of the other bidders will not try to match bidder 5's bid because $v_i < 100$. Thus it is again a best response for all parties and an equilibrium.

   While there is only one symmetric equilibrium there are many other asymmetric equilibria, and in some of them bidder 5 wins. Thus there can be a wide range of answers to this question. One example is $b_5 = \frac{100+v_1}{2}$ $b_j = v_6$.

Consider the following market. Demand for two goods is given by:

$$\begin{aligned} q_1 &= 144 - 3p_1 + 2p_2 \\ q_2 &= 144 - 3p_2 + 2p_1 \end{aligned}$$

and the firms have the same constant marginal cost: $c_1(q) = 24q_1$, $c_2(q) = 24q_2$.

1. First assume that these firms are Bertrand competitors.

   (a) Set up the objective function of one of the firms.

   $$\pi_1(p_1, p_2) = (144 - 3p_1 + 2p_2)(p_1 - 24)$$

   (b) Find the firm's best response.

   $$\begin{aligned} (144 - 3p_1 + 2p_2) - 3(p_1 - 24) &= 0 \\ \frac{1}{3}p_2 + 12 &= p_1 \end{aligned}$$

   (c) Find the equilibrium prices and quantities these two firms charge.

   $$\begin{aligned} \frac{1}{3}p + 12 &= p \\ 18 &= p \end{aligned}$$

   $$\begin{aligned} q &= 144 - 3(18) + 2(18) \\ &= 126 \end{aligned}$$

2. Now assume that these firms are Cournot competitors.

   (a) Verify that the inverse demand curves are:

   $$\begin{aligned} p_1 &= 144 - \frac{2}{5}q_2 - \frac{3}{5}q_1 \\ p_2 &= 144 - \frac{2}{5}q_1 - \frac{3}{5}q_2 \end{aligned}$$

$$\begin{aligned}
q_1 &= 144 - 3p_1 + 2p_2 \\
p_2 &= \frac{q_1 - 144 + 3p_1}{2} \\
q_2 &= 144 - 3\left(\frac{q_1 - 144 + 3p_1}{2}\right) + 2p_1 \\
p_1 &= 144 - \frac{2}{5}q_2 - \frac{3}{5}q_1
\end{aligned}$$

(b) Set up the objective function of one of the firms.

$$\pi_1(q_1, q_2) = \left(144 - \frac{2}{5}q_2 - \frac{3}{5}q_1\right) q_1 - 24q_1$$

(c) Find the firm's best response.

$$\begin{aligned}
\left(144 - \frac{2}{5}q_2 - \frac{3}{5}q_1\right) - \frac{3}{5}q_1 - 24 &= 0 \\
100 - \frac{1}{3}q_2 &= q_1
\end{aligned}$$

(d) Find the equilibrium quantities and prices these two firms charge.

$$\begin{aligned}
100 - \frac{1}{3}q &= q \\
75 &= q
\end{aligned}$$

$$\begin{aligned}
p &= 144 - \frac{2}{5}(75) - \frac{3}{5}(75) \\
&= 69
\end{aligned}$$

There are N voters who have positions that can be indicated by the numbers 1 through 7. The number of voter with each position is indicated in the table below:

1	2	3	4	5	6	7
12	0	0	0	4	5	4

1. What is the average position of these voters? What is the median position of these voters?

   There are $12 + 4 + 5 + 4 = 25$ voters at all. The average position of these voters is $\frac{1*12+5.4+6*5+7*4}{25} = \frac{90}{25} = 3.6$.The median position is the position of the middle (13th) voter which is 5.

2. Assume that voters always vote for the candidate who's position is closest to their own, and that there are two candidates. When they are indifferent the voters choose each candidate equally likely. Candidates only care about winning the election.

(a) For each position of candidate 2 find the best position (or positions) for candidate 1. Write your answer in the table below:

Position of candidate 2	1	2	3	4	5	6	7
Best positions for Candidate 1	2,3,4,5,6,7	3,4,5,6,7	4,5,6	5	5	5	3,4,5,6

Best responses have been listed above(It is necessary not to lose(or win, if possible) the election, So there exists more than one best responses in some cases). An answer does not have to list all the best responses, just one for each position of candidate 2.

(b) Find the Nash equilibrium of this game.
We have written the best responses of Player 1 but the best responses of Player 2 is also symmetric to these so all we need is to find pairs of positions (a, b) such that b is a best response of a and a is a best response of b and this holds only for (5,5)

Consider a model of Bertrand competition with differentiated demand. Firm one has a cost function $c_1(q_1) = 24q_1$ and firm 2 has the cost function $c_2(q_2) = 8q_2$. Demand for firm $i \in \{1, 2\}$ is given by (where $j \neq i$):

$$q_i = d_i(p_i, p_j) = 160 - 3p_i + 2p_j$$

1. Find the best response for firm 1 and firm 2 to every price of their opponent. (You may assume that the firm will not shut down.)
First we need write down the objective functions of both firms.

$$\begin{aligned} \pi_1(p_1, p_2) &= (p_1 - 24)(160 - 3p_1 + 2p_2) \\ \pi_2(p_1, p_2) &= (p_2 - 8)(160 - 3p_2 + 2p_1) \end{aligned}$$

To find the best responses we need to find the first order conditions of these objective functions.

$$\begin{aligned} \frac{d\pi_1}{dp_1} &= 0 \\ (160 - 3p_1 + 2p_2) - 3(p_1 - 24) &= 0 \\ 232 - 6p_1 + 2p_2 &= 0 \\ \frac{116 + p_2}{3} &= p_1 = p_1(p_2) \end{aligned}$$

and similarly

$$\begin{aligned} \frac{d\pi_2}{dp_2} &= 0 \\ (160 - 3p_2 + 2p_1) - 3(p_1 - 8) &= 0 \\ 184 - 6p_2 + 2p_1 &= 0 \\ \frac{92 + p_1}{3} &= p_2 = p_2(p_1) \end{aligned}$$

are the best response functions of both firms.

2. Find the Nash equilibrium of this game.

   Nash equilibrium occurs when both firms have their best responses at the same time.

$$\begin{aligned}\frac{116+\frac{92+p_1}{3}}{3} &= p_1\\ \frac{440+p_1}{9} &= p_1\\ 440 &= 8p_1\\ 55 &= p_1\end{aligned}$$

   And if $p_1 = 55$, $p_2 = \frac{92+55}{3} = 49$. Thus $(55, 49)$ is the Nash equilibrium of this game.

Consider the three following models of duopoly (two firms competing for profits in the same market.) In all cases firm one has a cost function $c_1(q_1) = 24q_1$ and firm 2 has the cost function $c_2(q_2) = 8q_2$.

1. The standard Bertrand model. Market demand is given by $D(p)$, where $D(24) > 0$ and the function is continuous and downward sloping. Firm i's demand is given by:

$$q_i = d_i(p_i, p_j) = \begin{cases} 0 & \text{if } p_i > p_j \\ \frac{1}{2}D(p_i) & \text{if } p_i = p_j \\ D(p_i) & \text{if } p_i < p_j \end{cases}$$

   and price must be in Kurus, which I will denote κ.

   (a) Find the best response to p_j assuming $(p_j - \kappa) q_i > c_i(q_i)$. (Hint: you should consider $\{p_j + \kappa, p_j, p_j - \kappa\}$ and you can assume that κ is *extremely* small and that p_j is less than the monopoly price for both firms.)

   The profits at each of the prices are:

$$\begin{aligned}\pi(p_j+\kappa) &= 0\\ \pi(p_j) &= \frac{1}{2}D(p_j)(p_j - c_i)\\ \pi(p_j-\kappa) &= D(p_j-\kappa)(p_j-\kappa-c_i)\end{aligned}$$

   Thus if $(p_j - \kappa) > c_i$ we can assume that

$$BR(p_j) = \begin{cases} p_j - \kappa & \text{if } p_j - \kappa - c_i > 0 \\ p_j & \text{if } p_j - \kappa - c_i \leq 0 \text{ and } p_j - c_i \geq 0 \\ p_j + \kappa & \text{if } p_j - c_i < 0 \end{cases}$$

   (b) *(4 Points)* Find a Nash equilibrium of this game. (While there are multiple Nash Equilibria all you have to do is find one and verify that it is an equilibrium.)

Clearly we must have $p_2 - \kappa - c_1 \leq 0$, and $p_2 - c_2 \geq 0$ because $c_1 > c_2$. In this range $p_1 = p_2 + \kappa$ is always a best response. Thus the equilibria are characterized by:

$$\begin{aligned} c_2 + \kappa &\leq p_2 \leq c_1 \\ p_1 &= p_2 + \kappa \end{aligned}$$

In this equilibrium firm 1 gets no demand, so they do not really care about the price, thus their action is a best response. Firm 2 gets all the demand, and they will make a positive profit. Thus these are all equilibria.

(c) Find the profit of the two firms in a Nash equilibria of this game.

$$\pi_1 = 0, \pi_2 = D\left(p_j\right)\left(p_j - c\right), \, p_j \in \left[c_2, \kappa + c_1\right]$$

2. Bertrand with differentiated demand. Demand for firm i is given by:

$$q_i = d_i\left(p_i, p_j\right) = 160 - 3p_i + 2p_j$$

since it is not helpful in this case price does not have to be in Kurus, (a price like 112.10450327 is fine.)

(a) Find the best response for firm 1 and firm 2 to every price of their opponent. (You may assume that the firm will not shut down.)
First we need write down the objective functions of both firms.

$$\begin{aligned} \pi_1(p_1, p_2) &= (p_1 - 24)(160 - 3p_1 + 2p_2) \\ \pi_2(p_1, p_2) &= (p_2 - 8)(160 - 3p_2 + 2p_1) \end{aligned}$$

To find the best responses we need to find the first order conditions of these objective functions.

$$\begin{aligned} \frac{d\pi_1}{dp_1} &= 0 \\ (160 - 3p_1 + 2p_2) - 3(p_1 - 24) &= 0 \\ 232 - 6p_1 + 2p_2 &= 0 \\ \frac{116 + p_2}{3} &= p_1 = p_1(p_2) \end{aligned}$$

and similarly

$$\begin{aligned} \frac{d\pi_2}{dp_2} &= 0 \\ (160 - 3p_2 + 2p_1) - 3(p_1 - 8) &= 0 \\ 184 - 6p_2 + 2p_1 &= 0 \\ \frac{92 + p_1}{3} &= p_2 = p_2(p_1) \end{aligned}$$

are the best response functions of both firms.

(a) Find the Nash equilibrium of this game.
Nash equilibrium occurs when both firms have their best responses at the same time.

$$\begin{aligned}\frac{116+\frac{92+p_1}{3}}{3} &= p_1\\ \frac{440+p_1}{9} &= p_1\\ 440 &= 8p_1\\ 55 &= p_1\end{aligned}$$

And if $p_1 = 55$, $p_2 = \frac{92+55}{3} = 49$. Thus $(55, 49)$ is the Nash equilibrium of this game.

(a) Find the quantities of both firms in equilibrium, and write down the profit and simplify as much as possible. (The profit might be too large to calculate easily, do not worry if you can not simplify it completely.)
Since $p_1 = 55$ and $p_2 = 49$ in the equilibrium, all we have to do is insert these values into the demand and objective functions.

$$\begin{aligned}q_i &= d_i(p_i, p_j) = 160 - 3p_i + 2p_j\\ \pi_1(p_1, p_2) &= (p_1 - 24)(160 - 3p_1 + 2p_2)\\ \pi_2(p_1, p_2) &= (p_2 - 8)(160 - 3p_2 + 2p_1)\end{aligned}$$

Thus, $q_1 = 160 - 3*55 + 2*49 = 93$ and $q_2 = 160 - 3*49 + 2*55 = 123$ are the quantities of firms.
And $\pi_1(55, 49) = 31 * 93 = 2883$, $\pi_2(55, 49) = 41 * 123 = 5043$ are the profits

3. Hotelling Linear City: There are N consumers located in a line, consumers buy from the firm that is closest to them (if both firms are equi-distant then they choose each firm with probability one half). Price is fixed (above the marginal cost of both firms) and firms compete by choosing location. Thus firms are essentially trying to maximize their demand. The number of consumers at each location is indicated in the table below:

0	1	2	3	4	5	6
3	8	1	3	2	9	9

(a) For each location of firm 2 find a best response (location) for firm 1. Write your answer in the table below:

Location of Firm 2	0	1	2	3	4	5	6
A Best Location for Firm 1	1	2	3	4	5	5	5
Average Demand of Firm 1	32	24	23	20	18	17,5	26

(b) Find the Nash equilibrium of this game.

Total demand is 35 and for every location of Firm 2 except 5, Firm 1 has a response which gives it more than half of the demand. Similar argument holds for Firm 2 as well. Therefore only case when best responses coincide is (5,5)

4. Which of these models is the best model of duopoly? Why is it better than the other ones? Notice that one could argue that any of these models is best, the points will be given for the argument not for "guessing right."

   One could start this argument with any of these models, but my favorite is Bertrand with differentiated demand. That produces a reasonable profit for both firms, and the prices are reasonable (above MC, etcetera.) It looks like a good model of oligopoly.

   However that model requires that the goods the two firms produce are different, i.e. not perfect substitutes. To see whether this is an equilibrium we should look at a Hotelling linear city model—where we interpret the distance between the two firms as how different their goods are. We find in equilibrium that both firms locate in the same place, so they produce perfect substitutes.

   This suggests that the best model is standard Bertrand oligopoly, but in that model the equilibrium has one firm not producing at all, and price below the marginal cost of that firm. This is ridiculous as a result, but then again the model has appeal.

   This suggests that the fact that the cross price elasticity is infinite is not acceptable, so this would lead us back to a model of Bertrand with differentiated demand.

3 Chapter 4—Mixed Strategy Equilibrium

1. Consider the Hawk/Dove game, the players are animals and each animal has two strategies. Either they can be *aggressive* (A) and hunt other animals to eat, or they can be *passive* (P) and eat plants. The payoffs of the game are:

		Player 2	
		A	P
Player 1	A	$0; 0$	$7; 5^{12}$
	P	$5; 7^{12}$	$4; 4$

   (a) Find the best responses for each player, you may mark them on the game but explain your reasoning carefully below. Half the points will be given for your explanation of your reasoning.

   If my opponent plays A $u(A, A) = 0 < 7 = u(P, a)$ *thus* P *is the best response.*

   If my opponent plays P $u(P, A) = 5 > 4 = u(P, P)$ *thus* A *is the best response.*

   In the table above I marked a 1 *or a* 2 *in the upper right hand corner of the strategy that is a best response.*

(b) Find the pure strategy Nash equilibria of this game.

They are (A, P) *and* (P, A)

(c) An equilibrium is *symmetric* if every agent uses the same strategy in equilibrium. Why might we want to find a symmetric equilibrium in this game?

If you think about the way this game plays out in nature it's not like there are actually two animals and so one can choose A and the other can choose P and be done with it. Instead the animals choose their strategy (A or P) and then randomly meet another animal and the payoffs are realized. So every species is playing against a random draw from the population, or they are playing against a symmrtric distribution over opponents. Thus it might be best to model the choice of their opponents as a random draw, and this means that every animal has the same expected utility and should use a symmetric strategy.

(d) Carefully show that there is no symmetric pure strategy Nash equilibrium of this game.

Assume that (A, A) *is this equilibrium, then as shown above* $BR(A) = P$ *and this is not a NE.*

Assume that (P, P) *is this equilibrium, then as shown above* $BR(P) = A$ *and this is not a NE.*

(e) Find the unique symmetric Nash equilibrium of this game.

Let ρ *be the probability that the opponent is aggressive, then:*

$$\begin{aligned} u(A, \rho) &= (1-\rho)(7) \\ u(P, \rho) &= \rho(5) + (1-\rho)4 = \rho + 4 \end{aligned}$$

$$\begin{aligned} u(A, \rho) &= u(P, \rho) \\ (1-\rho)(7) &= \rho(5) + (1-\rho)4 \\ &\rho = \frac{3}{8} \end{aligned}$$

is the symmetric mixed strategy Nash equilibrium. Obviously in reality we will have a fraction, ρ *in the population who will choose to be agressive before finding out who they are matched with, but in effect it will seem that they are playing a mixed strategy.*

2. About Mixed strategy Nash equilibria.

(a) Define a mixed strategy and explain how you would implement a mixed strategy. In other words if I told you to play an arbitrary mixed strategy what would you need to do?

A **mixed strategy** *is a random draw over the pure strategies of a player in a game. It is implimented by choosing the probabilities you use each strategy and then giving these probabilities to a third party, who will then randomize and choose your action.*

(b) Define a Mixed strategy Nash equilibrium for a game with a finite number of strategies and a finite number of players.

$\sigma^* \in \times_{i\in I}\Delta(A_i)$ is a mixed strategy equilibrium if for all i and $\sigma_i' \in \Delta(A_i)$ $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*)$. Or we can write as:

i. (rationality) *There exists a β_{-i}^i such that $\sigma_i^* \in \arg\max_{\sigma_i'\in\Delta(A_i)} u_i u_i(\sigma_i', \beta_{-i}^i)$.*

ii. (correct expectations) *for all i and j $\beta_j^i = \sigma_j^*$.*

(c) Assume that player 1 has two strategies, a and b, and that in a mixed strategy Nash equilibrium, σ^*, player 1 is supposed to play both a and b with positive probability. Prove that:

$$u_1(a, \sigma_{-i}^*) = u_1(b, \sigma_{-i}^*)$$

where σ_{-i}^* is the equilibrium mixed strategies of the other players. Also explain how this result is useful in finding mixed strategy Nash equilibria.

Assume not, and $u_1(a, \sigma_{-i}^*) > u_1(b, \sigma_{-i}^*)$ but then the mixed strategy $\tilde{\sigma}_i(a) = 1$ gives a strictly higher payoff because $u_i(\sigma_i^*, \sigma_{-i}^*) = \sigma_i^*(a) u_1(a, \sigma_{-i}^*) + (1 - \sigma_i^*(a)) u_1(b, \sigma_{-i}^*) < u_1(a, \sigma_{-i}^*) = u_i(\tilde{\sigma}_i, \sigma_{-i}^*)$.

This is useful because an implication of this result is that at σ_{-i}^* for all $\rho(a) \in [0,1]$

$\{\rho(a), 1-\rho(a)\} \in \arg\max_{\sigma_i'\in\Delta(A_i)} u_i u_i(\sigma_i', \sigma_{-i}^*)$, to see this realize obviously a or b is a best response and obviously if they both are then any convex combination of them is. Thus this proposition is what pins down $\rho^*(a)$.

3. About the definition of Nash equilibrium.

(a) Define a mixed strategy, and a (mixed strategy) best response.

A mixed strategy is a randomization over the pure strategies in the game. In other words for each strategy you want to play you choose a probability of playing that strategy, and then let someone else do the randomization. To be precise the set of mixed strategies is $\Delta(A_i) = \{p | \forall a_i \in A_i, p(a_i) \geq 0 \text{ and } \Sigma_i p(a_i) = 1\}$, a given mixed strategy is $\sigma_i \in \Delta(A_i)$.

Let $\sigma_{-i} \in \times_{j\neq i}\Delta(A_j)$, or a mixed strategy for all of the other players. Then $BR_i(\sigma_{-i}) \equiv \arg\max_{\sigma_i\in\Delta(A_i)} u_i(\sigma_i, \sigma_{-i})$

(a) Define a Mixed Strategy Nash equilibrium.

There are several ways to do it, let σ^* be a mixed strategy Nash equilibrium.

First of all if $\forall i, \forall \sigma_i \in \Delta(A_i)$ $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$.

Second of all $\forall i$ $\sigma_i^* \in BR_i(\sigma_{-i}^*)$, or σ^* is in the intersection of the best responses.

Finally there is the form I prefer:

i. (Rationality) *there is a* $\beta \in \times_{j \neq i} \Delta(A_j)$ *such that*

$$\sigma_i^* \in \arg \max_{\sigma_i \in \Delta(A_i)} u_i(\sigma_i, \beta)$$

ii. (Consistency) $\beta = \sigma_{-i}^*$.

4. Consider a public goods game. There are $I > 1$ people. Each person can decide whether to contribute (C) or not (N). If at least one person chooses C then everyone gets the public good and gets the same benefit of 18, however each person who contributes also has to pay the cost 2. (Thus that person's (or people's) utility is $18 - 2$.) If no one chooses to contribute then everyone gets zero.

(a) In a general I player game find the expected utilities of players from each action. Your formula should be written in terms of random events, like "the probability that the sun is shining." (Hint: think about the mixed strategy payoffs.)

$$\begin{aligned} EU_1(C, \cdot) &= 16 \\ EU_1(N, \cdot) &= 18 \Pr(\text{At least one other person contributes.}) \end{aligned}$$

(b) The two person game:

i. Draw a table representing this game in Normal form.

		Player 2	
		C	N
Player 1	C	$16; 16$	$16; \alpha^{12}$
	N	$18; 16^{12}$	$0; 0$

ii. Find the pure strategy best responses of both players and the pure strategy Nash equilibria. You may mark your answers on the table above but you will loose one point if you do not explain your notation below.
The best responses are marked with a 1 (2) for player 1 (2) on the table. The Nash equilibria are the boxes with a one and two in the upper right hand corner.

iii. These Nash equilibria are all asymmetric, why might we be interested in a symmetric Nash equilibrium for this game?
Because in many situations, like observing an accident, there is no clear way to coordinate between players. Consider, for example, an equilibrium where "the first who see it helps out." However then the first person could just pass by being absolutely confident that the next person will do it. Thus the only equilibrium that can withstand this coordination failure is a symmetric one, which in this case will be

iv. Find the symmetric (mixed strategy) Nash equilibrium. Let p be the probability that someone chooses C.

$$\begin{aligned} EU_1(C,p) &= 16 \\ EU_1(N,p) &= 18p \end{aligned}$$

$$\begin{aligned} 16 &= 18p \\ p &= \frac{8}{9} \end{aligned}$$

v. Find the probability that no one contributes in this mixed strategy Nash equilibrium.

$$\Pr(\text{No one contributes}) = (1-p)^2 = \left(1 - \left(1 - \frac{1}{9}2\right)\right)^2 = \frac{1}{18^2}2^2$$

(c) If $I = 3$

i. Find the symmetric mixed strategy Nash equilibrium. Let p be the probability that someone chooses C.

$$\begin{aligned} EU_1(C,p) &= 16 \\ EU_1(N,p) &= 18\left(p(1-p) + (1-p)p + p^2\right) \\ &= 18p(2-p) \\ &= 18\left(1-(1-p)^2\right) \end{aligned}$$

$$\begin{aligned} 16 &= 18\left(1-(1-p)^2\right) \\ 1 - \frac{1}{9} &= 1-(1-p)^2 \\ \frac{1}{9} &= (1-p)^2 \\ \sqrt{\frac{1}{9}} &= 1-p \\ p &= 1-\sqrt{\frac{1}{9}} \end{aligned}$$

ii. Find the probability that no one contributes in this mixed strategy Nash equilibrium, is it lower or higher than the probability you found in the two player game? Comment on the implications of this.

$$\Pr(\text{No one contributes}) = (1-p)^3 = \left(1 - \left(1 - \sqrt{\frac{1}{9}}\right)\right)^3 = \left(\frac{1}{9}\right)^{\frac{3}{2}}$$

(d) If $I > 3$ find the symmetric mixed strategy Nash equilibrium. Let p be the probability that someone chooses C. (Hint: You will only be able to find a formula for this answer.)

$$\begin{aligned} EU_1(C,p) &= 16 \\ EU_1(N,p) &= 18\left(1-(1-p)^{I-1}\right) \end{aligned}$$

$$\begin{aligned} 16 &= 18\left(1-(1-p)^{I-1}\right) \\ 1-\frac{1}{9} &= 1-(1-p)^{I-1} \\ \frac{1}{9} &= (1-p)^{I-1} \\ p &= 1-\left(\frac{1}{9}\right)^{\frac{1}{I-1}} \end{aligned}$$

5. About mixed strategy equilibria:

(a) Define a *mixed strategy equilibrium.* I will give partial credit for any answer that is approximately correct, but for full credit your answer must be precisely correct. *(Note there are several precisely correct answers.)*

My favored definition is that $\sigma^* \in \times_i \Delta(A_i)$ *is a mixed strategy NE if:*

i. For $\beta_i \in \times_{j\neq i}\Delta(A_j)$ $\sigma_i^* \in \arg\max_{\sigma_i\in\Delta(A_i)} u_i(\sigma_i, \beta_i)$

ii. $\beta_i = \sigma^* \backslash \sigma_i^*$ *(the mixed strategies of the opponents.)*

Another fine definition is that for all i and $\sigma_i \in \Delta(A_i)$, given $\sigma_{-i}^* = \sigma^* \backslash \sigma_i^*$

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(\sigma_i, \sigma_{-i}^*\right)$$

Another definition is that it is the intersection of best responses in mixed strategies. For example in a two player game it is:

$$\sigma_i^* \in BR_i\left(BR_{-i}\left(\sigma_i^*\right)\right) \ .$$

Notice that in both of these definitions (as clarified by question b) the fact that it is a weak inequality or only one element of the best responses to the best responses is important. For example in matching pennies:

$$\begin{aligned} BR_2\left(\frac{1}{2}\right) &= [0,1] \\ BR_1\left(BR_2\left(\frac{1}{2}\right)\right) &= [0,1] \end{aligned}$$

and $\frac{1}{2} \in [0,1]$ *the key thing is that for any* $p \neq \frac{1}{2}$, $BR_1(BR_2(p))$ *is either 1 or 0.*

(b) Let σ^* be a mixed strategy equilibrium, and σ^*_{-i} be the mixed strategies of the other players in the game, assume that a_i and $\hat{a}_i$ are played with strictly positive probability in σ_i^*, then what do we know about $U_i\left(a_i, \sigma^*_{-i}\right)$ and $U_i\left(\hat{a}_i, \sigma^*_{-i}\right)$? Why?

$U_i\left(a_i, \sigma^*_{-i}\right) = U_i\left(\hat{a}_i, \sigma^*_{-i}\right)$ *because if (say) a_i is lower player i can increase his payoff by using a strategy where $\sigma_i\left(a_i\right) = 0$.*

β	c	$p^* = \Pr(\alpha)$	$q^* = \Pr(A)$
2	2	$\frac{1}{4}$	$\frac{2}{3}$
3	6	$\frac{3}{4}$	$\frac{3}{4}$
6	3	$\frac{1}{3}$	$\frac{1}{4}$
4	4	$\frac{2}{3}$	$\frac{1}{3}$

6. Consider the following normal form game.

		Player 2		
		α	β	ψ
Player 1	A	8; 1	8; 0	3; 7
	B	1; 5	1; 7	2; 6
	C	2; 9	2; 6	6; 7

(a) Find all the best responses to pure strategies. You may mark them above but explain your notation below.

		Player 2		
		α	β	ψ
Player 1	A	$8; 1^1$	$8; 0^1$	$3; 7^2$
	B	$1; 5$	$1; 7^2$	$2; 6$
	C	$2; 9^2$	$2; 6$	$6; 7^1$

They are marked in the matrix above, the best responses for the row player are marked with a 1 in the upper right hand corner, the best responses for the column player are marked with a 2 in the upper right hand corner.

(b) Let p_α be the probability α is played in some mixed strategy, p_β be the probability β is played in the same mixed strategy, and write the payoffs of person 1 given this mixed strategy of player 2.

$U(A, p_\alpha, p_\beta) = 8p_\alpha + 8p_\beta + 3(1 - p_\alpha - p_\beta) = 3 + 5(p_\alpha + p_\beta)$
$U(B, p_\alpha, p_\beta) = 1p_\alpha + 1p_\beta + 2(1 - p_\alpha - p_\beta) = 2 - (p_\alpha + p_\beta)$
$U(C, p_\alpha, p_\beta) = 2p_\alpha + 2p_\beta + 6(1 - p_\alpha - p_\beta) = 6 - 4(p_\alpha + p_\beta)$

(c) Let q_A be the probability A is played in some mixed strategy, q_B be the probability B is played in the same mixed strategy, and write the payoffs of person 2 given this mixed strategy of player 1.

$U(\alpha, q_A, q_B) = 1q_A + 5q_B + 9(1 - q_A - q_B) = 9 - 8q_A - 4q_B$
$U(\beta, q_A, q_B) = 0q_A + 7q_B + 6(1 - q_A - q_B) = 6 - 6q_A + q_B$
$U(\psi, q_A, q_B) = 7q_A + 6q_B + 7(1 - q_A - q_B) = 7 - q_B$

(d) Find a cycle in the best responses and explain the cycle below.

		Player 2		
		α	β	ψ
	A	$8;1^1 \rightarrow$	$8;0^1$	$3;7^2 \downarrow$
Player 1	B	$1;5$	$1;7^2$	$2;6$
	C	$2;9^2 \uparrow$	$2;6$	$6;7^1 \leftarrow$

The direction you go from each element of the cycle is marked in the graph above.

(e) Assuming that actions that are not in the cycle you found in the last part of the question have zero probability, find the mixed strategy equilibrium. Afterwards calculate each person's expected utility from playing all of his or her actions in this mixed strategy equilibrium (including the action that is never played. Do not expect the answers to be integers.)

		p_α	$1-p_\alpha$
		α	ψ
q_A	A	$8;1^1 \rightarrow$	$3;7^2 \downarrow$
$1-q_A$	C	$2;9^2 \uparrow$	$6;7^1 \leftarrow$

$$
\begin{aligned}
8p_\alpha + 3(1-p_\alpha) &= 2p_\alpha + 6(1-p_\alpha) \\
p_\alpha &= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
1q_A + 9\,(1-q_A) &= 7q_A + 7\,(1-q_A) \\
q_A &= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
U_1(p_\alpha, q_A) &= \tfrac{1}{3}\tfrac{1}{4}8 + \tfrac{2}{3}\tfrac{1}{4}3 + \tfrac{1}{3}\tfrac{3}{4}2 + \tfrac{1}{3}\tfrac{3}{4}6 = \tfrac{38}{12} \\
U_2(p_\alpha, q_A) &= \tfrac{1}{3}\tfrac{1}{4}1 + \tfrac{2}{3}\tfrac{1}{4}7 + \tfrac{1}{3}\tfrac{3}{4}9 + \tfrac{1}{3}\tfrac{3}{4}7 = \tfrac{63}{12}
\end{aligned}
$$

7. In the following Normal form game:

	α	β	ψ	δ	ε
A	1; 2	5; 3	7; 4	4; 2	8; 3
B	0; 6	9; 8	6; 9	8; 10	3; 7
C	1; 13	2; 16	3; 6	0; 9	12; 12
D	2; 5	0; 4	2; 3	1; 4	2; 3
E	1; 4	6; 16	5; 6	10; 10	4; 5

(a) Find all of the best responses, you may mark them in the graph above.

	α	β	ψ	δ	ε
A	$1;2$	$5;3$	$\underline{7;4^{12}}$	$4;2$	$8;3$
B	$0;6$	$9;8^1 \rightarrow$	$6;9$	$8;10^2 \downarrow$	$3;7$
C	$1;13$	$2;16^2$	$3;6$	$0;9$	$12;12^1$
D	$\underline{2;5^{12}}$	$0;4$	$2;3$	$1;4$	$2;3$
E	$1;4$	$6;16^2 \uparrow$	$5;6$	$10;10^1 \leftarrow$	$4;5$

They are marked in the matrix above, the best responses for the row player are marked with a 1 in the upper right hand corner, the best responses for the column player are marked with a 2 in the upper right hand corner.

(b) Find all of the pure strategy Nash equilibria.

They are underlined in the matrix above.

(c) Find a cycle in the best responses.

The direction you go from each element of the cycle is marked in the graph above.

(d) Find a Nash equilibrium over the cycle you found in part c. (To be precise, only actions in the cycle have positive probability.)

the cycle is:

		q	$1-q$
		β	δ
p	B	$9;8^1 \rightarrow$	$8;10^2 \downarrow$
$1-p$	E	$6;16^2 \uparrow$	$10;10^1 \leftarrow$

$$\begin{aligned} 9q + 8(1-q) &= 6q + 10(1-q) \\ (9-6)q &= (10-8)(1-q) \\ q &= \frac{2}{5} \\ 8p + 16(1-p) &= 10p + 10(1-p) \\ 16 - 8p &= 10 \\ p &= \frac{3}{4} \end{aligned}$$

8. Consider the following strategic form game:

		Player 2			
		α	β	χ	δ
Player 1	A	6,0......	2,4......	1,2......	1,1......
	B	2,4......	3,3......	2,6......	2,2......
	C	3,4......	4,3......	1,2......	3,2......
	D	4,4......	6,2......	0,2......	2,2......

(a) Find the best responses of both players. You may mark them in the game above or write them down in the space below.

		Player 2			
		α	β	χ	δ
Player 1	A	$6{,}0^1 \rightarrow$	$2{,}4^2 \downarrow$	1,2	1,1
	B	2,4	3,3	$2{,}6^{12}$	2,2
	C	$3{,}4^2 \uparrow$	4,3	1,2	$3{,}2^1 \leftarrow$
	D	$4{,}4^2 \uparrow$	$6{,}2^1 \leftarrow$	0,2	2,2
		$q_\alpha = \frac{2}{3}$	$p_A = \frac{1}{3}$		

They are marked in the game above. An i in the upper right hand corner indicates that this is i's BR.

(b) Find the Nash equilibrium in pure strategies.

This is marked in the game above, it is the only square with both a 1 and a 2 in the upper right hand corner.

(c) Are there any cycles in best responses? If so mark them in the game above.

For completeness I showed where you would go from every best response. There is a cycle,

		q	$1-q$
		α	β
p	A	$6{,}0^1 \rightarrow$	$2{,}4^2 \downarrow$
$1-p$	D	$4{,}4^2 \uparrow$	$6{,}2^1 \leftarrow$

(d) Find the Mixed strategy Nash equilibrium of this game.

It is calculated below the game. The probability labeled $\{p_A, p_B, p_C, p_D\}$ *is the probability of that action, the same for the probability labeled* $\{q_\alpha, q_\beta, q_\chi, q_\delta\}$

$$\begin{aligned} u_1(A, \sigma_2) &= u_1(D, \sigma_2) \\ q_\alpha 6 + (1 - q_\alpha) 2 &= q_\alpha 4 + (1 - q_\alpha) 6 \\ q_\alpha &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} u_2(\alpha, \sigma_1) &= u_2(\beta, \sigma_1) \\ p_A 0 + (1 - p_A) 4 &= p_A 4 + (1 - p_A) 2 \\ p_A &= \frac{1}{3} \end{aligned}$$

(e) Which equilibrium is better for player 1? Which equilibrium is better for player 2?

In the game above the mixed strategy game gives a higher expected payoff to player 1, but the pure strategy game gives a higher payoff to player 2.

4 Chapter 9—Bayesian Games

1. Consider a standard auction with imperfect information. A bidder knows his own value v_i—$i \in \{1, 2, 3, ..., I\}$ but all he knows about other bidder's

values are that each one is distributed uniformly over $[0,1]$, so the cumulative distribution function for bidder j's value $(j \neq i)$ is $F(v_j) = v_j$. The winner will always be the person who bid the highest, and if person i wins and has to pay p then their utility function is $v_i - p$. If person i does not win they get zero. If several people bid the same amount they are equally likely to win.

(a) Second Price Auction: In this auction the high bidder has to pay the second highest bid.

i. Show that bidding your own value, $b_i = v_i$, is a weakly dominant strategy in this game.

I will be done if I can show that for any $\tilde{b}_i \neq v_i$ there is always a situation where $b_i = v_i$ is a better strategy, and never a case where it is worse. First of all if there is no $b_j \in \left[\tilde{b}_i, b_i\right]$ or $b_j \in \left[b_i, \tilde{b}_i\right]$ then they will give the same outcome. If $b_j > \max\left(b_i, \tilde{b}_i\right)$ then the bidder will loose. if $b_j < \max\left(b_i, \tilde{b}_i\right)$ then the bidder will win and pay the highest of such b_j. Since $b_i = v_i$ the bidder will get at least zero profits in this case and not regret winning. Thus assume $\tilde{b}_i < b_i$ and that there is a $b_j \in \left[\tilde{b}_i, b_i\right]$. In this case the winner will lose the auction win they could have won and gotten at least zero profit, thus they will at least weakly wish that $b_i = v_i$. In the opposite case the bidder will win the auction and get at least negative utility. Thus $b_i = v_i$ is always a weakly better strategy than any $\tilde{b}_i \neq v_i$.

ii. Show that there is an equilibrium where $b_4 = 1$ and $b_j = 0$ for every $j \in \{1, 2, 3, ..., I\} \setminus 4$.

Since $b_4 \geq v_j$ no bidder will want to win the auction at a price of 1, thus $b_j = 0$ is optimal. Since $b_j = 0 \leq v_4$ bidder 4 will not regret bidding one.

(b) The First Price Auction: In this auction the high bidder pays the amount they bid. Assume throughout that they use a symmetric strategy of the form $b_i = \alpha v_i + \beta$, where $\alpha > 0$.

i. Write down the objective function of a bidder in this auction.

Since $v_i\tilde{\ }U(0,1)$ $b_j\tilde{\ }U(\beta, \alpha+\beta)$ and the cumulative distribution function of b_j is $F(b_j) = \frac{b_j - \beta}{\alpha + \beta - \beta}$ thus the objective function is:

$$\Pr\left(b_i \geq \max_{j \neq i} b_j\right)(v_i - b_i)$$
$$\Pr(b_i \geq b_j)^{I-1}(v_i - b_i)$$
$$F(b_i)^{I-1}(v_i - b_i)$$
$$\left(\frac{b_i - \beta}{\alpha}\right)^{I-1}(v_i - b_i)$$

ii. Prove that if $v_i = 0$ then $b_i = 0$, or that $\beta = 0$.
Assume $\beta > 0$, then a bidder who's value is 0 when they win will win $-\beta$, thus $\beta \leq 0$. Assume $\beta < 0$ then in the case where they win they will win $-\beta\frac{1}{I}$ because they will win only when everyone else has the value of zero and in this case since they are using a symmetric strategy they will win with probability $\frac{1}{I}$, whereas if they bid $\frac{\beta}{1.5}$ they will win with $-\frac{\beta}{1.5} > -\beta\frac{1}{2} \geq -\beta\frac{1}{I}$ thus $\beta = 0$.

iii. Find the first order condition of the objective function. (Assume that $\beta = 0$.)

$$\left(\frac{b_i}{\alpha}\right)^{I-1}(v_i - b_i)$$

$$\frac{(I-1)}{\alpha}\left(\frac{b_i}{\alpha}\right)^{I-2}(v_i - b_i) - \left(\frac{b_i}{\alpha}\right)^{I-1} = 0$$

iv. Find the formula for the bid, and verify that it has the linear form $b_i = \alpha v_i$.

$$\begin{aligned}
(I-1)\frac{b_i^{I-2}}{\alpha^{I-1}}(v_i - b_i) - \frac{b_i^{I-1}}{\alpha^{I-1}} &= 0 \\
(I-1)(v_i - b_i) - b_i &= 0 \\
\frac{(I-1)}{I}v_i &= b_i
\end{aligned}$$

2. Consider a Bertrand game of differentiated demand. The demand for firm 1 and 2 is:

$$\begin{aligned}
q_1 &= 54 - p_1 + \frac{1}{2}p_2 \\
q_2 &= 54 - p_2 + \frac{1}{2}p_1
\end{aligned}$$

$$\begin{aligned}
q_1 &= a - \gamma p_1 + \gamma\tau p_2 \\
q_2 &= a - \gamma p_2 + \gamma\tau p_1
\end{aligned}$$

The costs of firm 2 are $c_2(q_1, q_2) = 0$, the costs of firm 1 are $c_1(q_1, q_2) = 0$ with probability ρ and $c_1(q_1, q_2) = cq_1$ with probability $1 - \rho$, where $\rho \in (0, 1)$. Firm 1 knows her costs, firm two does not.

(a) Set up the two objective functions for firm 1.

$$\max_{p_1^h}\left(54 - p_1^h + \frac{1}{2}p_2\right)\left(p_1^h - 60\right)$$

$$\max_{p_1^l}\left(54 - p_1^l + \frac{1}{2}p_2\right)p_1^l$$

(b) *(6 points)* Find the two best response formulas for firm 1.

$$\begin{aligned}
\left(54 - p_1^h + \frac{1}{2}p_2\right) - \left(p_1^h - c\right) &= 0 \\
54 + 60 - 2p_1^h + \frac{1}{2}p_2 &= 0 \\
54 + 60 + \frac{1}{2}p_2 &= 2p_1^h \\
\frac{1}{4}p_2 + 57 &= \frac{54 + 60 + \frac{1}{2}p_2}{2} = p_1^h
\end{aligned}$$

$$\begin{aligned}
\left(54 - p_1^l + \frac{1}{2}p_2\right) - p_1^l &= 0 \\
54 - 2p_1^l + \frac{1}{2}p_2 &= 0 \\
54 + \frac{1}{2}p_2 &= 2p_1^l \\
\frac{1}{4}p_2 + 27 &= \frac{54 + \frac{1}{2}p_2}{2} = p_1^l
\end{aligned}$$

(c) *(3 points)* Find the expectation of p_1 for firm 2, it should be a function of p_2.

$$\begin{aligned}
E\left(p_1\right) &= \rho p_1^h + \left(1-\rho\right)p_1^l \\
&= \rho\frac{54 + \frac{1}{2}p_2}{2} + \left(1-\rho\right)\frac{54 + c + \frac{1}{2}p_2}{2} \\
&= \frac{1}{2}60\left(1-\rho\right) + \frac{1}{2}\frac{1}{2}p_2 + \frac{1}{2}54 = \frac{1}{4}p_2 - 30\rho + 57
\end{aligned}$$

: :

(d) Set up the objective function for firm 2, be sure to include the fact that p_1 is a random variable.

$$\max_{p_2}\left(54 - p_2 + \frac{1}{2}E\left(p_1\right)\right)p_2$$

(e) Find best response formula for firm 2.

$$\begin{aligned}
\left(54 - p_2 + \frac{1}{2}E\left(p_1\right)\right) - p_2 &= 0 \\
54 - 2p_2 + \frac{1}{2}E\left(p_1\right) &= 0 \\
54 + \frac{1}{2}E\left(p_1\right) &= 2\gamma p_2 \\
\frac{54 + \frac{1}{2}E\left(p_1\right)}{2} &= p_2
\end{aligned}$$

(f) Find the Bayesian Nash equilibrium prices.

$$
\begin{aligned}
p_2 &= \frac{1}{2}\left(54+\frac{1}{2}\left(\frac{1}{2}c(1-\rho)+\frac{1}{2}\frac{1}{2}p_2+\frac{1}{2}\frac{54}{1}\right)\right)\\
p_2 &= \frac{1}{4}60\frac{1}{2}+\frac{1}{2}\frac{54}{1}+\frac{1}{4}54\frac{1}{2}+\frac{1}{4}*\frac{1}{2}^2 p_2-\frac{1}{4}60\frac{1}{2}\rho\\
p_2\left(1-\frac{1}{4}*\frac{1}{2}^2\right) &= \frac{1}{4}60\frac{1}{2}+\frac{1}{2}\frac{54}{1}+\frac{1}{4}54\frac{1}{2}-\frac{1}{4}60\frac{1}{2}\rho\\
p_2 &= \frac{\frac{1}{4}60\frac{1}{2}+\frac{1}{2}\frac{54}{1}+\frac{1}{4}54\frac{1}{2}-\frac{1}{4}60\frac{1}{2}\rho}{\left(1-\frac{1}{4}*\frac{1}{2}^2\right)}\\
p_2 &= \frac{54}{\left(2-\frac{1}{2}\right)}+\frac{60\frac{1}{2}}{4-\frac{1}{2}^2}(1-\rho)=44-8\rho
\end{aligned}
$$

$$
\begin{aligned}
\frac{54+60+\frac{1}{2}p_2}{2} &= p_1^h\\
\frac{54+60+\frac{1}{2}\left(\frac{54}{\left(2-\frac{1}{2}\right)}+\frac{60\frac{1}{2}}{4-\frac{1}{2}^2}(1-\rho)\right)}{2} &= p_1^h\\
\frac{54}{\left(2-\frac{1}{2}\right)}+\frac{1}{8-2*\frac{1}{2}^2}\left(4*60-60*\frac{1}{2}^2\rho\right) &= p_1^h\\
\frac{54+\frac{1}{2}p_2}{2} &= p_1^l\\
\frac{54+\frac{1}{2}\left(\frac{54}{\left(2-\frac{1}{2}\right)}+\frac{60\frac{1}{2}}{4-\frac{1}{2}^2}(1-\rho)\right)}{2} &= p_1^l\\
\frac{54}{2-\frac{1}{2}}+\frac{60*\frac{1}{2}^2}{8-2*\frac{1}{2}^2}(1-\rho) &= p_1^l
\end{aligned}
$$

3. Cafe Nero has finally come to Ankara! And being the sensible people they are they decided to open near to Bilkent first, specifically in the Real shopping center. Unfortunately this causes you a problem, you usually go to flirt with that special someone at Starbucks, and now you are afraid that you might want to go to Cafe Nero instead.

   Your strategy set is N—go to Care Nero—and S—go to Starbucks. Both of you get a utility of 1 from being at the same coffee shop with that other person and 2 from going to Starbucks, however your utility of going to Cafe Nero is unknown. Each of you knows your own u_i, but all you know about the other person's u_j is that it is uniformly distributed over $[0,7]$, thus it has the cumulative distribution function of $F(u_j)=\frac{u_j}{7}$ $(j \neq i)$.

   (a) Draw a normal form game that represents this situation. You should have the values u_1 and u_2 in the payoffs of this table.

	N	S
N	$u_1 + 1; u_2 + 1$	$u_1; 2$
S	$2; u_2$	$3; 3$

(b) Prove that there is no pure strategy Nash equilibrium of this game, i.e. you can not choose N all the time and you can not choose S all the time. (You should consider different values of u_i and u_j).
If $u_1 < 1$ then $u_1(N, N) < u_1(S, N)$ and S is a dominant strategy. Likewise if $u_1 > 3$ then $u_1(S, S) < u_1(N, S)$ and N is a dominant strategy.

(c) Write down a cut off strategy that you may want to use in this game.

$$S(u_i) = \begin{cases} N & \text{if} \quad u_i \geq u_i^* \\ S & \text{else} \end{cases}$$

(d) Given this cutoff strategy write down the expected payoffs of using the strategies N and S.

$$\begin{aligned} Eu(N) &= \left(1 - F\left(u_j^*\right)\right)(u_i + 1) + F\left(u_j^*\right) u_i = 1 - F\left(u_j^*\right) + u_i \\ Eu(S) &= \left(1 - F\left(u_j^*\right)\right) 2 + F\left(u_j^*\right) 3 = 2 + F\left(u_j^*\right) \end{aligned}$$

(e) Find the equilibrium cut off strategies in this game, you may assume they are symmetric.

$$\begin{aligned} 1 - F(u^*) + u^* &= 2 + F(u^*) \\ u^* &= 1 + 2F(u^*) \\ u^* &= 1 + 2\frac{u^*}{7} \\ au^* &= 7 + 2u^* \\ u^* &= \frac{7}{7-2} = \frac{7}{5} \end{aligned}$$

4. Consider two firms that are simultaneously deciding whether or not to enter an industry. If they stay out (O) they get zero. If they enter (E) they have to pay a fixed cost f_i $(i \in \{1, 2\})$. This cost is private information for firm $i \in \{1, 2\}$, all the other firm knows is that it is distributed uniformly over $[0, 12]$. The cumulative distribution function of f is $G(f) = \frac{f}{12}$. If only one firm enters it earns monopoly revenue of 8 so it's total profit is $8 - f_i$, if both firms enter then they earn duopoly revenue of 4, so their total profits are $4 - f_i$.

(a) Write down a normal form game with the payoffs above, note that f_i will vary and should be a part of your payoffs.

		Firm 2: E	Firm 2: O
Firm 1	E	$4 - f_1; 4 - f_2$	$8 - f_1; 0$
	O	$0; 8 - f_2$	$0; 0$

(b) Prove that there is no pure strategy equilibrium in this normal form game. Where by pure strategy I mean that one person takes the same action for all f_i.

Assume that in equilibrium one firm always plays E, then the highest payoff they can get is $8 - f_i$, however if $f_i > 8$(which is always possible since $12 > 8$) *this payoff is negative, thus they should choose O.*

Alternatively assume one firm always player O, then if they switch to E the lowest payoff they can get is $4 - f_i$ and if $f_i < 4$ (which is possible since $4 > 0$) *they should change their strategy.*

(c) What is a cutoff strategy? What type of cutoff strategy do you think people will use in this game?

A "cutoff strategy" is a strategy where there is some key variable like f_i and a key value of that variable, let me denote it f^* such that if $f_i \leq f^*$ one action is taken and if $f_i > f^*$ another is taken.

In this game it is obvious that the strategy will be something like:

$$A(f_i) = \begin{cases} E & \text{if} \quad f_i \leq f^* \\ O & \text{if} \quad f_i > f^* \end{cases}$$

(d) Find a symmetric equilibrium in cutoff strategies.

$$\begin{aligned} U_i(O, f^*) &= 0 \\ U_i(E, f^*) &= G(f^*)(4 - f_i) + (1 - G(f^*))(8 - f_i) \\ &= (8 - f_i) + G(f^*)(L - f_i - (8 - f_i)) \\ &= (8 - f_i) - G(f^*)(8 - 4) \\ &= (8 - f_i) - \frac{f^*}{12}(8 - 4) \end{aligned}$$

when $f_i = f^*$:

$$\begin{aligned} U_i(O, f^*) &= U_i(E, f^*) \\ 0 &= (8 - f^*) - \frac{f^*}{12}(8 - 4) \\ f^* &= \frac{8}{8 - 4 + 12} 12 = 6 \end{aligned}$$

5. Consider the following public goods game. There are three people in this society. Donating costs $c \geq 0$, if one person donates then the public good is produced, giving each person a benefit of 4. The utility of a person is their benefits minus their costs. If someone donates this is denoted D, if she or he does not this is denoted N.

   (a) Assume that $0 < c < 4$ and that c is common knowledge and the same for all parties. Some answers will depend on c.

i. Find the pure strategy best responses.

$$\begin{aligned}
u_1(D,D,D) &= 4-c \\
u_1(N,D,D) &= 4 \\
u_1(D,D,N) &= 4-c \\
u_1(N,D,N) &= 4 \\
u_1(D,N,D) &= 4-c \\
u_1(N,N,D) &= 4 \\
u_1(D,N,N) &= 4-c \\
u_1(N,N,N) &= 0
\end{aligned}$$

$$\begin{aligned}
BR_1(D,D) &= BR_1(N,D) = BR_1(D,N) = N \\
BR_1(N,N) &= D
\end{aligned}$$

by symmetry the answer is

$$BR = \begin{cases} D & \text{if no one else is donating} \\ N & \text{else} \end{cases}$$

ii. Find the pure strategy Nash equilibria.
In all the pure strategy equilibria one person donates and everyone else does not. There are three of them.

iii. Notice that none of these pure strategy Nash equilibria are *symmetric*, why might we be interested in a symmetric Nash equilibrium?
Because asymmetric equilibria require some sort of implicit pre-play communication, in many cases (like the reporting a crime game) the game is played too infrequently for implicit pre-play communication to work. Thus we might want to look at a symmetric equilibrium.

iv. Find a symmetric Nash equilibrium, and then calculate the probability that the public good will be provided in that symmetric Nash equilibrium. *(Hint: this probability is not one.)*
Let $p = \Pr(D)$, *then if you donate you will get* b *always*

$$U(D,p,p) = 4-c$$

if you do not donate then you only get b *if at least one other donates*

$$\begin{aligned}
\Pr(\text{at least one other donation}) &= 1 - \Pr(\text{no other donations}) \\
p^2 + 2p(1-p) &= 1 - (1-p)^2 \\
-p(p-2) &= -p(p-2)
\end{aligned}$$

$$U(N,p,p) = \left(1-(1-p)^2\right)4$$

in equilibrium these are equal:

$$\begin{aligned}
\left(1-(1-p)^2\right)4 &= 4-c \\
c &= 4-\left(\left(1-(1-p)^2\right)4\right) \\
c &= 4(1-p)^2 \\
\left(\frac{c}{4}\right)^{\frac{1}{2}} &= 1-p \\
p &= 1-\left(\frac{c}{4}\right)^{\frac{1}{2}}
\end{aligned}$$

The probability the public good will be provided is:

$$\begin{aligned}
\Pr(\text{at least one donates}) &= 1-\Pr(\text{no one donates}) \\
3p(1-p)^2+3p^2(1-p)+p^3 &= 1-(1-p)^3 \\
p\left(p^2-3p+3\right) &= p\left(p^2-3p+3\right) \\
\Pr(\text{at least one donates}) &= 1-(1-p)^3 \\
&= 1-\left(\sqrt{\frac{c}{4}}\right)^3
\end{aligned}$$

(b) Assume that c is distributed over $[0,16]$ with the cumulative distribution function $F(c) = \frac{1}{4}\sqrt{c}$; and that each person knows her or his personal value of c, but not the other players.

i. Prove that there is no pure strategy equilibrium.
In such a pure strategy equilibrium someone would have to always donate, but if $c > 4$ this person would not want to donate, since this happens with positive probability there is no such equilibrium.

ii. Find a symmetric equilibrium in cutoff strategies.
Since someone with c higher than 4 will never donate the strategy will be:

$$A(c) = \begin{cases} D & \text{if } c \le c^* \\ N & \text{else} \end{cases}$$

thus the probability someone donates is $F(c^*)$, from above we know that this is equal to p.

$$\begin{aligned}
U(D,F(c^*),F(c^*)) &= 4-c \\
U(N,F(c^*),F(c^*)) &= \left(1-(1-F(c^*))^2\right)4
\end{aligned}$$

$$\begin{aligned}
U(D,F(c^*),F(c^*)) &= U(N,F(c^*),F(c^*)) \\
4-c^* &= \left(1-(1-F(c^*))^2\right)4 \\
(1-F(c^*))^2 &= \frac{c^*}{4} \\
1-\sqrt{\frac{c^*}{4}} &= F(c^*) \\
1-\sqrt{\frac{c^*}{4}} &= \sqrt{\frac{c^*}{16}} \\
\frac{16}{9} &= c^*
\end{aligned}$$

iii. Find the probability that the public good will be provided in this equilibrium.
again, using a similar reasoning as above:

$$\begin{aligned}
\Pr(\text{at least one donates}) &= 1-(1-p)^3 = 1-(1-F(c^*))^3 \\
&= 1-\left(1-\left(1-\sqrt{\frac{c^*}{4}}\right)\right)^3 \\
&= 1-\frac{1}{8}(c^*)^{\frac{3}{2}} \\
&= 1-\left(\sqrt{\frac{c^*}{4}}\right)^3 = 1-\left(\sqrt{\frac{\frac{16}{9}}{4}}\right)^3 \\
&= 1-\frac{8}{27}=\frac{19}{27}
\end{aligned}$$

6. Consider a market for Bertrand with differentiated demand. The demand curves of the firms are symmetric:

$$\begin{aligned}
q_1 &= 108-2p_1+p_2 \\
q_2 &= 108-2p_2+p_1.
\end{aligned}$$

However while firm 1 has the cost function of $c_1(q_1)=15q_1$ all it knows about firm 2's cost is that it is $c_2(q_2)=20q_2$ with probability ρ and $c_2(q_2)=0$ with probability $1-\rho$. Firm 2 knows its own costs and the costs of firm 1.

(a) Set up all the objective functions for the two firms.

$$\begin{aligned}
\pi_1 &= (108-2p_1+Ep_2)(p_1-15) \\
\pi_2^h &= (108-2p_2+p_1)(p_2-20) \\
\pi_2^l &= (108-2p_2+p_1)p_2
\end{aligned}$$

(b) Find the best responses for all types of all firms.

$$
\begin{aligned}
& \pi_1 \\
(108 - 2p_1 + Ep_2) - 2(p_1 - 15) &= 0 \\
p_1 &= \frac{1}{2}15 + \frac{1}{2}\frac{108}{2} + \frac{1}{4}Ep_2 = \frac{1}{4}Ep_2 + \frac{69}{2} \\
& \pi_2^h \\
(108 - 2p_2 + p_1) - 2(p_2 - 20) &= 0 \\
p_2^h &= \frac{1}{2}20 + \frac{1}{2}\frac{108}{2} + \frac{1}{4}p_1 = 37 + \frac{1}{4}p_1 \\
& \pi_2^l \\
(108 - 2p_2 + p_1) - 2p_2 &= 0 \\
p_2^l &= \frac{1}{2}\frac{108}{2} + \frac{1}{4}p_1 = 27 + \frac{1}{4}p_1
\end{aligned}
$$

(c) Find the price firm 1 chooses in equilibrium.

$$
\begin{aligned}
p_1 &= \frac{1}{2}15 + \frac{1}{2}\frac{108}{2} + \frac{1}{4}Ep_2 \\
p_1 &= \frac{1}{2}15 + \frac{1}{2}\frac{108}{2} + \frac{1}{4}\left(\rho\left(\frac{1}{2}20 + \frac{1}{2}\frac{108}{2} + \frac{1}{4}p_1\right) + (1-\rho)\left(\frac{1}{2}\frac{108}{2} + \frac{1}{4}p_1\right)\right) \\
p_1 &= \frac{1}{4 - \frac{1}{2}^2}\left(30 + 108 + \frac{1}{2}20\rho + \frac{108}{2}\frac{1}{2}\right) = \frac{8}{3}\rho + 44
\end{aligned}
$$

(d) Find the prices firm 2 may choose in equilibrium.

$$
\begin{aligned}
p_2^h &= \frac{1}{2}20 + \frac{1}{2}\frac{108}{2} + \frac{1}{4}p_1 \\
&= \frac{1}{2}20 + \frac{1}{2}\frac{108}{2} + \frac{1}{4}\left(\frac{1}{4 - \frac{1}{2}^2}\left(30 + 2\frac{108}{2} + \frac{1}{2}20\rho + \frac{108}{2}\frac{1}{2}\right)\right) \\
&= \frac{1}{4 - \frac{1}{2}^2}\left(40 + \frac{1}{2}15 + 108 - \frac{1}{2}\frac{1}{2}^2 20 + \frac{1}{2}\frac{1}{2}^2 20\rho + \frac{108}{2}\alpha\right) = \frac{2}{3}\rho + 48 \\
p_2^l &= \frac{1}{2}\frac{108}{2} + \frac{1}{4}p_1 \\
&= \frac{108}{4} + \frac{1}{4}\left(\frac{1}{4 - \frac{1}{2}^2}\left(30 + 2\frac{108}{2} + \frac{1}{2}20\rho + \frac{108}{2}\frac{1}{2}\right)\right) \\
&= \frac{1}{4 - \frac{1}{2}^2}\left(1 + 2\frac{108}{2} + \frac{1}{2}\frac{1}{2}^2 20\rho + \frac{108}{2}\frac{1}{2}\right) = \frac{2}{3}\rho + 38
\end{aligned}
$$

7. Consider the Prisoner's Dilemma with costs of betrayal. Two criminals (and friends) are caught committing a robbery. During the robbery a murder was committed but the police have no evidence the criminals committed it.

   The police tell each criminal that if only one of them confesses (C) to the murder then that person will go free, with no prison sentence, the other will be convicted of the murder and the robbery. If both confess then both will be convicted of murder. If neither confess (both choose quiet, Q) then both of them will be convicted for the robbery.

   The difference between this and the standard prisoner's dilemma is that now a criminal feels bad if he confesses. Confessing will cost that criminal c_i. This is private information to person $i \in \{1, 2\}$, all that the other person knows is that it is distributed uniformly over $[0, 5]$, with a cumulative distribution function of $F(c) = \frac{c}{5}$. The payoffs in the stage game are the number of years spent in prison minus the cost of confessing (if applicable).

		Player 2	
		C	Q
Player 1	C	$-20 - c_1; -20 - c_2$	$-c_1; -22$
	Q	$-22; -c_2$	$-2; -2$

   assume throughout that both players will use a cutoff strategy.

   (a) Find the expected payoff of player 1 from playing C and Q for any cutoff strategy of player 2.

   I will guess that the player 2 will follow the strategy:

$$S_2 = \begin{cases} C & \text{if} \quad c_2 \leq c_2^* \\ Q & \text{if} \quad c_2 > c_2^* \end{cases}$$

$$\begin{aligned} u_1(C, c_2^*) &= F(c_2^*)(-a - c_1) + (1 - F(c_2^*))(-c_1) = -F(c_2^*)a - c_1 \\ u_1(Q, c_2^*) &= F(c_2^*)(-(a+b)) + (1 - F(c_2^*))(-b) = -F(c_2^*)a - b \end{aligned}$$

   (b) Prove that there is no equilibrium where both parties always choose one of the two strategies.

$$\begin{aligned} u_1(C, c_2^*) &> u_1(Q, c_2^*) \\ -F(c_2^*)a - c_1 &> -F(c_2^*)a - b \\ c_1 &< b \end{aligned}$$

   since $c_1 < b$ and $c_1 > b$ both have positive probability it is not possible for either player to always play one or the other, independent of what the other player does.

(c) Find all the equilibria where both parties sometimes confess.

The formula for c_1^* is:

$$
\begin{aligned}
-F(c_2^*)\,a - c_1^* &= -F(c_2^*)\,a - b \\
c_1^* &= b
\end{aligned}
$$

this is independent of what player 2 does, and by symmetry we can see that $c_2^* = b$. Thus this is the only NE. Notice that since it is independent of $F(\cdot)$ this is also the NE for any $F(\cdot)$ where the support of c contains an open set around b.

8. Consider the following two normal form games, in this question only analyze pure strategies.

Game α

Player 1 \ Player 2	L	C	R
U	$5,3$	$0,4$	$6,5$
M	$4,1$	$5,3$	$7,0$
D	$3,1$	$0,0$	$0,2$

Game β

Player 1 \ Player 2	O	T	A
U	$6,5$	$6,3$	$0,2$
M	$0,1$	$10,0$	$5,2$
D	$7,0$	$4,1$	$0,2$

(a) Find the best responses and Nash equilibria in both games. You may mark the best responses on the graph above. Write down the equilibrium *strategies* below.

(M, C) and (M, A).

(b) Now assume that player 1 does not know which game he is playing, instead he thinks he is playing game α with probability p and game β with probability $1-p$.

i. For each strategy of player 1, $s_1 \in \{U, M, D\}$, find the best response of player 2.

$$
\begin{aligned}
BR_2(U) &= (R, O) \\
BR_2(M) &= (C, A) \\
BR_2(D) &= (R, A)
\end{aligned}
$$

ii. Explain why if we want to find a pure strategy Nash equilibrium we can ignore player 2's strategies that are not best responses to some $s_1 \in \{U, M, D\}$.

Because we need that $BR_1(BR_2(s_1)) = s_1$ so if something is not a best response to s_1 we don't need to consider it. It will never be used in equilibrium.

iii. For each of the strategies of player 2 found in part *b.i.* of this question and all values of p find the expected utility of player 1 of playing each action against that strategy in the table below.

Across the top you should write down each of the three strategies you found in the last part of the question, and then below write the expected value of each action against that strategy.

If $S_2 =$	(R,O)	(C,A)	(R,A)
$U_1(T,S_2)$	γ	0	$p\gamma$
$U_1(M,S_2)$	$p(\gamma+1)$	5	$p(\gamma+1)+(1-p)5$
$U_1(B,S_2)$	$(1-p)(\gamma+1)$	0	0
	....	....	....

iv. For all values of p find the Nash equilibria.
First of all $(M,(C,A))$ is always a NE, and we can also see that for any p B is never a best response to (R,A), instead M is, so that is not a NE. Thus the only candidate that is left is $(T,(R,O))$ and T is a best response to (R,O) if:

$$\begin{aligned} p(\gamma+1) &\leq \gamma \\ p &\leq \frac{\gamma}{\gamma+1} \\ (1-p)(\gamma+1) &\leq \gamma \\ p &\geq \frac{1}{\gamma+1} \end{aligned}$$

so it is a NE if $p \in \left[\frac{1}{\gamma+1}, \frac{\gamma}{\gamma+1}\right]$.

9. Assume that two firms are Bertrand competitors with differentiated products. Each firm's demand curves are:

$$\begin{aligned} q_1 &= 60 - p_1 + \frac{2}{3}p_2 \\ q_2 &= 60 - p_2 + \frac{2}{3}p_1 \end{aligned}$$

firm 2's costs are $c_2(q) = 32pq_2$, firm 1's costs are $c_1(q) = 0$ with probability p and $c_1(q) = 96q_1$ with probability $1-p$.

(a) Find the best response of firm 1 to firm 2's price if $c_1(q) = 96q_1$.

$$\left(60 - p_1 + \frac{2}{3}p_2\right) - (p_1 - 96) = 0$$

$$p_1^c = \frac{1}{2}\left(156 + \frac{2}{3}p_2\right)$$

(b) Find the best response of firm 1 to firm 2's price if $c_1(q) = 0$.

$$p_1\left(60 - p_1 + \frac{2}{3}p_2\right)$$

$$\left(60 - p_1 + \frac{2}{3}p_2\right) - p_1 = 0$$
$$p_1^0 = \frac{1}{2}\left(60 + \frac{2}{3}p_2\right)$$

(c) Find the best response of firm 2 to firm 1's price.

$$(p_2 - 32p)\left(60 - p_2 + \frac{2}{3}p_1\right)$$
$$-(p_2 - 32p) + \left(60 - p_2 + \frac{2}{3}p_1\right) = 0$$
$$p_2 = \frac{1}{2}\left(60 + 32p + \frac{2}{3}p_1\right)$$

(d) Find the equilibrium prices of the two firms. *Hint—there is something peculiar about the equilibrium.*

$$\begin{aligned} pp_1^0 + (1-p)\,p_1^c &= p\left(\frac{1}{2}\left(60 + \frac{2}{3}p_2\right)\right) + (1-p)\left(\frac{1}{2}\left(156 + \frac{2}{3}p_2\right)\right) \\ &= \frac{1}{2}\left(60 + 96 + \frac{2}{3}p_2 - 96p\right) \end{aligned}$$

$$p_2 = \frac{1}{2}\left(60 + \frac{1}{2}96p\frac{2}{3} + \frac{2}{3}\left(\frac{1}{2}\left(60 + 96 + \frac{2}{3}p_2 - 96p\right)\right)\right)$$

$$\begin{aligned} p_2 &= \frac{1}{4 - \frac{4}{9}}\left(120 + 40 + 96\frac{2}{3} + 96p\frac{2}{3} - 96p\frac{2}{3}\right) \\ p_2 &= \frac{1}{\left(4 - \frac{4}{9}\right)}\left(\left(2 + \frac{2}{3}\right)60 + 64\right) \end{aligned}$$

$$\begin{aligned} p_1^0 &= \frac{1}{2}\left(60 + \frac{2}{3}\left(\frac{1}{\left(4 - \frac{4}{9}\right)}\left(120 + 40 + 96\frac{2}{3}\right)\right)\right) \\ &= \frac{1}{2\left(4 - \frac{4}{9}\right)}\left(\left(\frac{2}{3} + 2\right)120 + 96\frac{2}{3}\right) \end{aligned}$$

$$\begin{aligned} p_1^c &= \frac{1}{2}\left(60 + 60 + 40\left(\frac{1}{\left(4 - \frac{4}{9}\right)}\left(120 + 40 + 96\frac{2}{3}\right)\right)\right) \\ &= \frac{1}{\left(4 - \frac{4}{9}\right)}\left(\left(\frac{2}{3} + 2\right)60 + 192\right) \end{aligned}$$

The weird thing about the equilibrium is that it is independent of p. Yes, truly a cooked result but heh, I thought it might make things easier.

10. There are two firms that are both working to invent the *brain chip*, which will allow people to type into computers by only thinking about it. If company $i \in \{1, 2\}$ invents the brain chip it will cost them C_i, which is known only to that company. They simultaneously decide whether to *invent (I)* or *not invent (N)*. If they choose I they invent and sell the brain chip and it costs them C_i. All that the other firm and the government knows about C_i is that it is distributed independently and uniformly over $[0, 60]$, this means that $F(C) = \Pr(C_i \leq C) = \frac{C}{60}$. Each firm knows how much it will cost them to invent the brain chip, the government does not. The government is deciding whether or not to grant the inventor a monopoly (patent) on the brain chip.

   (a) Assume that the government does not grant a monopoly to the inventor. Then if either one invents the brain chip both companies will get 12 because they will both produce the brain chip and sell it.

      i. Prove that there can be no equilibrium where one firm always invents and the other does not.
      Assume that firm 1 is always supposed to invent, but then with positive probability $C_i > 12$ thus they will not want to invent. Then the other firm will invent if, for example, $C_i = 0$.

      ii. Find the payoff of a representative firm if they invent the brain chip.

$$\Pi(I) = 12 - C$$

      iii. Find the payoff to a representative firm if they do not invent the brain chip.

$$\Pi(N) = F(C^*)\,12 = \left(\frac{C^*}{60}\right)12$$

      iv. Find the symmetric equilibrium.

$$\begin{aligned} \left(\frac{C}{60}\right)12 &= 12 - C \\ C^{np} &= 10 \end{aligned}$$

   (b) Assume that the government does grant a monopoly to the inventor. Then if one firm invents it that firm will get 48, if both firms invent it then both firms will get 12. Notice that since firms decide simultaneously whether to invent it or not it is possible for them both to invent it at the same time.

      i. Prove that there can be no equilibrium where one firm always invents and the other does not.
      Assume that firm 1 is always supposed to invent, but then with positive probability $C_i > 12$ thus they will not want to invent. Then the other firm will invent if, for example, $C_i = 0$.

ii. Find the payoff of a representative firm if they invent the brain chip.

$$\begin{aligned}\Pi(I) &= (1 - F(C^*))\,48 + F(C^*)\,12 - C \\ &= \left(1 - \frac{C^*}{60}\right) 48 + \frac{C^*}{60} 12 - C\end{aligned}$$

iii. Find the payoff to a representative firm if they do not invent the brain chip.

$$\Pi(N) = 0$$

iv. Find the symmetric equilibrium.

$$\left(1 - \frac{C}{60}\right) 48 + \frac{C}{60} 12 - C = 0$$

$$C^p = 30$$

(c) Assume that the government only cares about the probability that the brain chip is invented. Find this probability for each case (a and b) and find out whether the government should issue monopolies to inventors (patents) or not.

11. Consider the following two normal form games, in this question only analyze pure strategies.

Game α

		Player 2: L	Player 2: R
Player 1	T	$9;3$	$0;5$
	M	$0;1$	$3;2$
	B	$6;8$	$2;4$

Game β

		Player 2: O	Player 2: A
Player 1	T	$3;2$	$0;1$
	M	$0;6$	$9;4$
	B	$2;7$	$6;8$

(a) Find the best responses and Nash equilibria in both games. You may mark the best responses on the graph above.

Game α

		Player 2: L	Player 2: R
Player 1	T	$9;3^1$	$0;5^2$
	M	$0;1$	$3;2^{12}$
	B	$6;8^2$	$2;4$

Game β

		Player 2: O	Player 2: A
Player 1	T	$3;2^{12}$	$0;1$
	M	$0;6^2$	$9;4^1$
	B	$2;7$	$6;8^2$

The Nash equilibrium has both a 1 and a 2 *in the upper right hand corner.*

(b) Now assume that player 1 does not know which game he is playing, instead he thinks he is playing game α with probability p and game β with probability $1 - p$.

i. For all values of p find the payoffs of player 1 and fill out the following table :

If $S_2 =$	(L,O)	(L,A)
$U_1(T,S_2)$	$9p+3(1-p)=6p+3$	$9p+0(1-p)=9p$
$U_1(M,S_2)$	$0p+0(1-p)=0$	$0p+9(1-p)=9-9p$
$U_1(B,S_2)$	$6p+2(1-p)=4p+2$	$6p+6(1-p)=6$

If $S_2 =$	(R,O)	(R,A)
$U_1(T,S_2)$	$0p+3(1-p)=3-3p$	$0p+0(1-p)=0$
$U_1(M,S_2)$	$3p+0(1-p)=3p$	$3p+9(1-p)=9-6p$
$U_1(B,S_2)$	$2p+2(1-p)=2$	$2p+6(1-p)=6-4p$

ii. For all values of p find the Nash equilibria.

The first thing to note is that player 2 will only use the strategies (R,O) and (L,A), (R,O) is the best response to T or M and (L,A) is the best response to B.
T is a best response to (R,O) if

$$\begin{aligned} 3-3p &\geq 2 \\ p &\leq \frac{1}{3} \end{aligned}$$

M is a best response if

$$\begin{aligned} 3p &\geq 2 \\ p &\geq \frac{2}{3} \end{aligned}$$

thus these are equilibria if p is in one of those regions.
B is a best response to (L,A) if:

$$\begin{aligned} 6 &\geq 9-9p \\ p &\geq \frac{1}{3} \\ 6 &\geq 9p \\ p &\leq \frac{2}{3} \end{aligned}$$

thus these are Nash equilibria in these cases. To be clear, if $p \leq \frac{1}{3}$ then (R,O) T is a Nash equilibrium, if $\frac{1}{3} \leq p \leq \frac{2}{3}$ then B (L,A) is a Nash equilibrium, if $\frac{2}{3} \leq p$ then (R,O) M is a Nash equilibrium.

(c) For which values of p do both players prefer that player 1 does not know what game he is playing?

Both players prefer (L,A) B to either (M,R) or (T,O) so both players prefer it if $p \in \left(\frac{1}{3}, \frac{2}{3}\right)$. Furthermore notice that 1's expected payoff when 2 plays (R,O) is always lower than 3, the payoff they get if they know the game.

12. Consider a second-price auction with a binding is reservation price r.

    In a second price auction there are I bidders who have values identically and independently distributed on $[\underline{v}, \bar{v}]$, the person who bids the most wins the item and pays the second highest bid. If i has the value v_i and wins at the price of p her utility is $v_i - p$, otherwise it is zero.

    If there is a reservation price then the winner must always pay at least r, and they must bid more than r to win. We say that r is *binding* if there is a strictly positive probability that any bidder's value is strictly lower than r.

    Prove that if there is a binding reservation price then there is no equilibrium where a given person (1 for example) *always* wins the auction, regardless of the values of the bidders.

    The equilibrium that a given person, say i-th agent always wins the auction regardless of the values of the bidders if the following holds: for arbitrary $i \in I$, $b_i = \bar{v}$ *and* $b_k = r$ *for all* $k \in I \backslash i$. *Recall that bidding his value is optimal for agent i.*

    Since there is a positive probability that $v_i < r$ *player i will not always be willing to bid this amount. Therefore assume that* $v_k > r$ *for some* $k \in I \backslash i$, *then by bidding* $b_k = v_k$ *with strictly positive probability he will win the auction and make a strictly positive surplus. Likewise if* $v_k < r$ *then it is better to bid* $b_k = v_k$ *since then there is no probability they win at the price of* r. *Thus this can not be an equilibrium.*

13. A consumer is buying a car of unknown value. He knows that the car is equally likely to be worth $k * 1000$ for $k \in \{1, 2, 3, 4, 5, 6, 7\}$. If the car is worth $k * 1000$ to the seller then the buyer values the car at $\beta * k * 1000$. The seller knows the value of the car. The buyer makes a take it or leave it offer.

    (a) Given that the buyer offers p, which sellers will be willing to sell their car?

    The sellers that $p \geq k * 1000$ *will be willing to sell their cars since* $u_s = p - k * 1000$ *if* $p \geq k * 1000$ *and zero otherwise.*

    (b) Given that the buyer offers p, what is the average value of the car the buyer will receive?

    One can show that $E(U|p) = \frac{k+1}{2}\beta(1000)$, *as the following calculations show:*

$$\begin{array}{ccc}
0 & \text{if} & p \leq 1000 \\
1000\beta & \text{if} & 1000 < p \leq 2000 \\
1000\beta\left(\frac{1}{2} + \frac{1}{2} * 2\right) = 1500\beta & \text{if} & 2000 \leq p < 3000 \\
1000\beta\left(\frac{1}{3} + \frac{1}{3} * 2 + \frac{1}{3} * 3\right) = 2000\beta & \text{if} & 3000 \leq p < 4000 \\
1000\beta\left(\frac{1}{4} + \frac{1}{4} * 2 + \frac{1}{4} * 3 + \frac{1}{4} * 4\right) = 2500\beta & \text{if} & 4000 \leq p < 5000 \\
1000\beta\left(\frac{1}{5} + \frac{1}{5} * 2 + \frac{1}{5} * 3 + \frac{1}{5} * 4 + \frac{1}{5} * 5\right) = 3000\beta & \text{if} & 5000 \leq p < 6000 \\
1000\beta\left(\frac{1}{6} + \frac{1}{6} * 2 + \frac{1}{6} * 3 + \frac{1}{6} * 4 + \frac{1}{6} * 5 + \frac{1}{6} * 6\right) = 3500\beta & \text{if} & 6000 \leq p < 7000 \\
1000\beta\left(\frac{1}{7} + \frac{1}{7} * 2 + \frac{1}{7} * 3 + \frac{1}{7} * 4 + \frac{1}{7} * 5 + \frac{1}{7} * 6 + \frac{1}{7} * 7\right) = 4000\beta & \text{if} & 7000 \leq p
\end{array}$$

(c) For **all** values of β find the optimal amount for the buyer to offer.

First of all it should be clear that $p = k{*}1000$ *for* $k \in \{0,1,2,3,4,5,6,7\}$ *and we can see from above that if* $p = k * 1000$ *then* $E(U|p) = \frac{k+1}{2}\beta(1000) = 500\beta(k+1)$. *So the consumer should choose k to maximize:*

$$500\beta(k+1) - k1000 = 500\beta - 1000k + 500k\beta$$

So if $\beta < 2$ *the optimal* $p = 0$. *If* $\beta \geq 2$ *then the optimal* $p = 7000$.

(d) Why is this called a model of *adverse selection*?

Because for any price p offered by the buyer, only the cars that worth less than or equal to that price will be sold by the sellers.

14. Assume that there is a monopolist who makes a take it or leave it offer to a consumer. The monopolist has no value to the good, the buyer has a value which is distributed on the interval $[v_l, v_h]$ with a CDF of $F(\cdot)$ and a PDF of $f(\cdot)$.

(a) Assume first of all that the probability of v_l is p and the probability of v_h is $1-p$.

i. Prove that the monopolist will never make an offer that is neither v_l nor v_h,

If monopolist makes the offer some v such that $v_l < v < v_h$,

If the customer has valuation is v_l *the monopolist's payoff is 0 but he could have gained a positive payoff if he has offered* v_l, *and if customer has valuation* v_h *monopolist will have expected utility (1-p)v whereas he could have gained (1-p)*v_h. *Thus strategy v is dominated and the monopolist will never make an offer that is neither* v_l *nor* v_h

ii. Find the critical value of p such that the monopolist will make an offer of v_l.

$p{*}0 + (1-p)v_h \leq p * v_l + (1-p)v_l = v_l$

the critical value is attained at the equality $(1-p)v_h = v_l$, *thus* $p = \frac{v_h - v_l}{v_h}$

iii. Consider changing $v_l + b$ and v_h to $v_h + b$, find the critical value of p such that the offer is v_l as a function of b.

$p{*}0 + (1-p)(v_h+b) \leq p * (v_l+b) + (1-p)(v_l+b) = v_l + b$

$v_h + b - (v_l + b) = p(v_h + b)$

$p = \frac{v_h - v_l}{v_h + b}$

iv. Prove that it is always Pareto efficient for the monopolist to sell the good to all types of consumers.

If monopolist offers v_l *both the customer is better off for all types and as we have seen above for the values of p that is greater than or equal to the critical value we have found above it is better for the monopolist to offer* v_l.

(b) Now characterize the optimal price for the monopolist for general $F(\cdot)$. (This should be found in terms of the first order condition.)

15. Consider a market where the value of the object is known to the seller but not to the buyer. If the object has a value of w to the seller than it has a value of $v_i w$ to buyer i. Assume that there are a large number of buyers and sellers so the price will be determined so that the demand equals the expected supply.

    The value to the seller is distributed uniformly over the range $[0, 10]$, and there are 100 sellers, each of whom has one unit to sell. Quantity is divisible.

    If Q units are sold on the market then the marginal unit will be sold to someone for whom $v_i = 252 - 5Q$.

    (a) Given that the market price is P, find the expected value of the marginal buyer.

$$\begin{aligned} E(w|w \leq P) &= \int_0^P \frac{zf(z)\,dz}{F(P)} = \int_0^P \frac{z\frac{1}{10}dz}{\frac{P}{10}} = \frac{1}{2}P \\ E(U_i|P) &= (252 - 5Q)\frac{1}{2}P \end{aligned}$$

    (b) Given that the market price is P, find the expected supply.

$$\begin{aligned} Q &= 100F(P) \\ &= 100\left(\frac{P}{10}\right) = 10P \end{aligned}$$

    (c) Find the equilibrium quantity that will be sold in this market and the price at which it will be sold.

$$\begin{aligned} P &= (252 - 5Q)\frac{1}{2}P \\ Q &= 50 \\ v_i &= 252 - 5 * 50 = 2 \\ 50 &= 10P \\ P &= 5 \end{aligned}$$

16. Consider a bank which is lending to investors. All investors need 1000 YTL, and they will have a return of 5 YTL per lira invested with probability γ, and 0 with probability $1 - \gamma$. Find a condition on γ such that a bank can afford to offer loans to the investors.

$$\begin{aligned} 1000 * 5 * \gamma + (1-\gamma) * 1000 * 0 &\geq 1000 \\ 5000\gamma &\geq 1000 \\ \gamma &\geq \frac{1}{5} \end{aligned}$$

17. Consider a model of Firm-Union bargaining. The revenue the firm will generate in the next year is 14, and this is known to all parties. However the firm does not know whether the union is strong or weak. If the union is strong and is offered any wage below 8 it will go on strike and both parties will get a payoff of zero. If it is weak then it will accept any offer above 2, if it is offered a lower wage then it will go on strike. The probability that the union is strong is q. *Assume that both parties will always accept any offer if they are indifferent between accepting it and rejecting it.*

    (a) First consider a model where the firm makes a take it or leave it offer of w, and the union can either accept it or go on strike.

    i. Find the best response of both types of unions to a wage offer of w.
    If w^r is the reservation wage then they accept when $w \geq w^r$

    ii. Write down the profit of the firm when they offer 8 and when they offer 2.

$$\begin{aligned} \Pi(8) &= 14 - 8 = 6 \\ \Pi(2) &= (1-q)(14-2) = (1-q)\,12 \end{aligned}$$

    iii. If $q = \frac{1}{4}$ what wage will the firm offer? Will the union go on strike?

$$\begin{aligned} \Pi(8) &= 6 \\ \Pi(2) &= (1-q)\,12 = \left(1 - \frac{1}{4}\right) 12 = 9 \end{aligned}$$

    the firm will offer 2 since it gives a higher profit and strong firms will go on strike.

    iv. If $q = \frac{3}{4}$what wage will the firm offer? Will the union ever go on strike?

$$\begin{aligned} \Pi(8) &= 6 \\ \Pi(2) &= \left(1 - \frac{3}{4}\right) 12 = 3 \end{aligned}$$

    so the firm will offer 8 and the union will never go on strike.

v. Someone points out that the fact that the union goes on strike proves that is strong, and therefore the firm should offer anyone who goes on strike a high wage. This does not work, why not?
This requires you to think outside of the box. However in this model the cost of striking is the same to both types of unions, so any time the strong union will strike for a high wage the weak union will strike as well. They may as well ask the union if it is strong or not.

(b) Now consider a model where both the firm and the union simultaneously declare a wage. If the wage offered by the firm is higher than the wage offered by the union then the union gets the wage the firm offers, otherwise the firm goes on strike.

i. Find an equilibrium where the firm never strikes.
If both types of unions demand a wage higher than 8 but less than 14 then the firm will match this since it at least gives them zero profits. Thus the union will never strike.

ii. Show that the best equilibrium for the firm when $q = \frac{1}{4}$ and $q = \frac{3}{4}$ are as you found above in parts a.iii and a.iv.
This really requires no proof. In the previous part they were able to select the wage to maximize the firms profit, so it has to be the best of all possible equilibria.

iii. Describe the full set of equilibria when $q = \frac{1}{4}$ and $q = \frac{3}{4}$.
If $q = \frac{3}{4}$ *then the equilibrium wage is* $14 \geq w \geq 8$. *If* $q = \frac{1}{4}$ *we also can have* $8 \geq w \geq 2$ *and the firm will strike one fourth of the time.*

18. Consider a second-price auction. In this auction there is one indivisible good that is awarded to each of the highest bidders with equal likelihood (notice if there is only one high bidder then it is given to that bidder with certainty). Each bidder submits one sealed bid and the price the highest bidder pays is the highest of all the other bids. The values of the bidders are distributed on $[v_l, v_h]$ where $\infty > v_h > v_l > 0$. For simplicity assume that bids must be in Kurus, that value of every bidder is in Kurus, and that one Kurus is very small relative to the value of each bidder (or v_l).

(a) Prove that it is weakly dominant for a bidder to bid his value. (In other words always an optimal strategy regardless of the strategies of the other players.)
In such an auction the winnings of i *if he wins will be* $v_i - b_{(2)} | b_{(2)} \leq b_i$ *so if* $b_i \leq v_i$ *this will always be weakly positive. Furthermore the probability that other's bids are less than* b_i *is always weakly increasing in* b_i, *thus it is optimal to increase* b_i *until* $b_i = v_i$.

(b) Find an equilibrium where bidder 1 always wins.
If $b_1 = v_h$ *then every other bidder can make at most zero profit by bidding* v_j, *thus they can set* $b_j = v_l$ *(or 0) without any change in*

their payoffs. Thus there is an equilibrium where $b_1 = v_h$, $b_j \leq v_l$ *for* $j \neq 1$.

(c) In many auctions there is a *reservation price*, or a price at which the object is only sold when the highest bid is (weakly) higher than the reservation price, and the price is always at least the reservation price. We say that a reservation price (r) is *binding* if $F(r) > 0$, or there is a positive probability that the value of any bidder is strictly below this price, and thus also a positive probability that the value of every bidder is below this price.

Show that if there is a binding reservation price then there is no equilibrium where bidder 1 always wins if his value is higher than the reserve price. *(Hint: Think about the cases where* $v_1 < r$ *and* $v_1 = r + \varepsilon$*).*

Considering the case where $v_1 = r + \varepsilon$ *for small* ε*, clearly we must have* $b_j \leq r$*. However now consider the case where* $v_1 < r$*. In this case some* $j \neq 1$ *will win the auction. Thus any bidder who has* $v_j > r$ *will at least want to set* $b_j = r$*. Now if person k bids r then with positive probability he will loose the auction when he wants to win it, thus he will bid* $b_k = r + \kappa$ *(one Kurus). Thus we can not have an equilibrium where* $b_j \leq r$ *and we have a contradiction. (Notice we could proceed by this logic to establish that the winner will not have* $v_k > b_k$ *as long as there is a positive probability someone else will bid* $b_j = b_k$*, or the winner will bid* $v_k = b_k$*.)*