ECON 439

Practice Questions—Extensive Form Games

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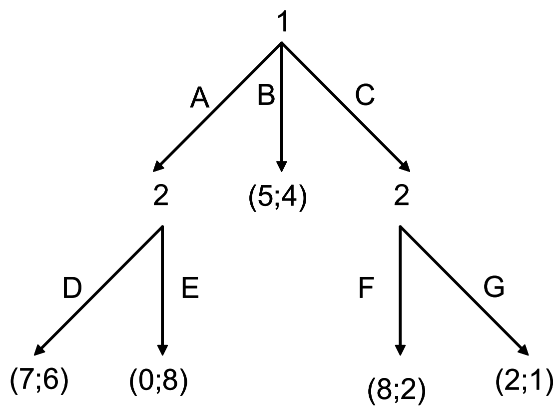
These questions are supposed to help you prepare for exams and quizzes, they are not to be turned in. Answers will be posted before the relevant exam.

1 Chapter 5—Sequential or Extensive form Games with Perfect Information.

1. Consider the following sequential game



   (a) Find the set of strategies of both players.
   $P1 : \{A, B, C\}$
   $P2 : \{(D, F), (D, G), (E, F), (E, G)\}$

   (b) Find the Subgame Perfect equilibrium strategies. You may mark your answers on the game above but write the strategy below.
   $BR(A) = E$, $BR(C) = F$, $BR(\emptyset) = C$.
   $\{C, (E, F)\}$

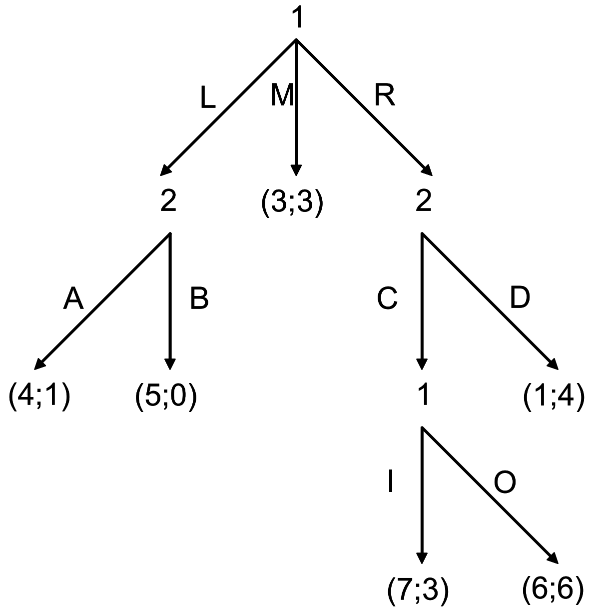
   (c) Transform this game into a Strategic Form Game and draw the game table below.

	(D,F)	(D,G)	(E,F)	(E,G)
A	$7;6$	$7;6^1$	$0;8^2$	$0;8^2$
B	$5;4^2$	$5;4^2$	$5;4^2$	$5;4^{12}$
C	$8;2^{12}$	$2;1$	$8;2^{12}$	$2;1$

   (d) Find a Nash equilibrium that is not a Subgame Perfect equilibrium and yet gives the same outcome.
   $\{C, (D, F)\}$ *since the action after A gives a strictly lower payoff it is irrelevant whether player 2 chooses D or E. Of course E is the only best response at that point.*

(e) Find a Nash equilibrium strategy that give a different outcome, and explain how there is an *empty threat* in this Nash equilibrium.
$\{B, (E, G)\}$ *in this equilibrium player 2 is threatening to do something stupid if player 1 chooses* C. G *gives both parties a strictly worse outcome. If player 1 believes this empty threat then he will decide to play* B. *This gives player 2 a higher payoff, and so he would like it if his empty threat was believed.*

2. Consider the following sequential game or extensive form game of perfect information.



(a) Find the best response at each decision node (or after every non-terminal history). *You may mark them above but explain your notation below or you will loose 2 points.*

$$\begin{aligned} BR(R, C) &= I \\ BR(R) &= D \\ BR(L) &= A \\ BR(\emptyset) &= L \end{aligned}$$

(b) Write down all of each player's strategies.
$S_1 = \{(L, I), (L, O), (M, I), (M, O), (R, I), (R, O)\}$
$S_2 = \{(A, C), (A, D), (B, C), (B, D)\}$

(c) Find the subgame perfect equilibrium strategies. You will get no points for merely writing down the tactics used in equilibrium.
$(L, I)(A, D)$

(d) Using this game explain why it is important to write down the equilibrium strategies instead of the tactics or the outcome.

One could just write down L, A, which would specify the outcome, but from these actions we can not verify that this is an equilibrium. If 2 used C instead of D then this would not be optimal, and if 1 used O instead of I this would justify 2 using C.

3. Consider the following game:

		Player 2 α	β	BR_2	
Player 1	A	$10; -1^{1}$	$2; 8^{\ 2}$	$2; 8$	β
	B	$8; 11^{2}$	$3; 4$	$8; 11^{1}$	α
	C	$1; 7$	$6; 11^{12}$	$6; 11$	β

(a) Find the best responses of both players. You may mark them in the table above but you will automatically loose one point if you do not explain your notation below.

Boxes that are a best response for player 1 have a 1 in the upper right hand corner, those for P2 have a 2 in the upper right hand corner.

(b) Find the Nash equilibrium.

In each game it is the box with a 1 and a 2 in the upper right hand corner.

(c) Now transform this into a sequential game where player one chooses between A, B, and C and then after observing what player one has done player two chooses between α and β. Draw this game in the space below.

This is difficult to do, thus I will only describe it. First there should be a decision node with three branches, one labeled A, one labeled B, and one labeled C. After each one of these decision nodes there should be a decision node with two branches, one labeled α and one labeled β. The payoffs after each decision node should correspond to the payoffs in the game above.

(d) *(5 points)* Find all the strategies of both players. You will be given one half point per strategy (rounded up). Be clear in your notation.

$$(A, B, C)\left\{\begin{array}{l} (\alpha(A), \alpha(B), \alpha(C)), (\beta(A), \alpha(B), \alpha(C)), (\alpha(A), \beta(B), \alpha(C)), (\alpha(A), \alpha(B), \beta \\ (\beta(A), \beta(B), \beta(C)), (\alpha(A), \beta(B), \beta(C)), (\beta(A), \alpha(B), \beta(C)), (\beta(A), \beta(B), \alpha \end{array}\right.$$

(e) Solve the sequential game by backward induction. You may mark your answers on the game above but you will automatically loose one point if you do not explain your notation below.

The best response for 2 are the actions corresponding to the payoffs in the column BR_2 added to the games above. The strategy of player 1 is the action corresponding to the box in this column with a 1 in the upper right hand corner. This will be the SPE outcome.

(f) Write down the Subgame Perfect equilibrium strategies of both players. *Hint: When I say strategies I mean strategies, zero points will be awarded for incomplete strategies.*

The outcome in part e pins down player 1's strategy, the strategy of player 2 are the actions that result in the column BR_2

(g) Explain by example why it is important to write down the full strategies, not the tactics that will be used on the equilibrium path.

I will answer this for the game:

		Player 2		
		α	β	BR_2
Player 1	A	$10; -1^1$	$2; 8\ ^2$	$2; 8$
	B	$8; 11^2$	$3; 4$	$8; 11^1$
	C	$1; 7$	$6; 11^{12}$	$6; 11$

The equilibrium strategies are $(B), (\beta(A), \alpha(B), \beta(C))$ *but the equilibrium tactics are* (B, α) *if I just write these down this leaves unclear what happens if one chooses A or C, if player two is using the strategy* $\alpha(A)$ *instead of* $\beta(A)$ *the strategy B would no longer be optimal. Thus writing down tactics does not make clear whether the actions taken are optimal or not.*

(h) What is an *empty threat*? Find a Nash equilibrium that is not a Subgame Perfect Equilibrium of the sequential game and explain how empty threats are used in this strategy.

An empty threat is an action that will not be taken if a choice is ever required, but if the other person believes this action will be taken then the person who makes the threat will never have to take the action.

An example for the game:

		Player 2	
		α	β
Player 1	A	$10; -1^1$	$2; 8\ ^2$
	B	$8; 11^2$	$3; 4$
	C	$1; 7$	$6; 11^{12}$

is $(\beta(A), \beta(B), \beta(C))$ *the best response to this strategy is* C, *resulting in the Nash equilibrium of* (β, C). *There may be others, but this is an example.*

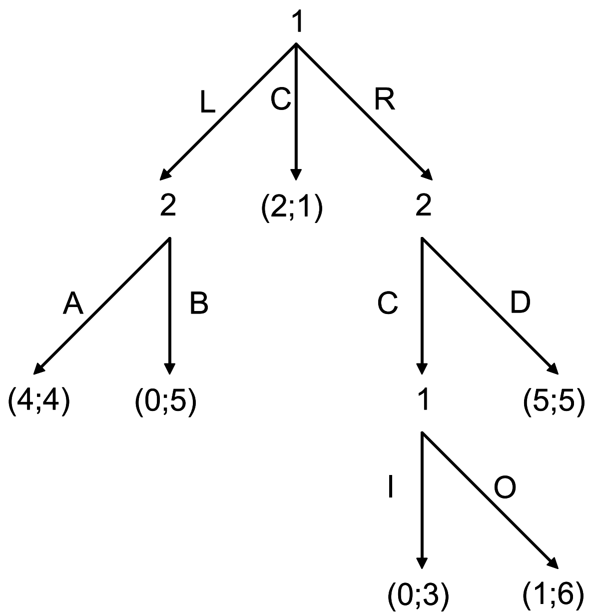
4. Give a precise definition of *first mover's advantage* and prove that it is true.

Definition: *Take a simultaneous two player game and make it sequential so that one player moves first. The player who moves first will get a better payoff in the Subgame Perfect equilibrium of the sequential game than they got in any pure strategy Nash equilibrium of the simultaneous game.*

Proof: *The player who moves second will have exactly the same best responses in both situations. In the simultaneous game they are answering the question "if the other person does X," in the sequential game they are answering the question "given the other person did X." These questions will have the same answer.*

Thus the person who moves first can choose to take an action they took in a pure strategy Nash equilibrium of the simultaneous game, the other person will best respond and that will be the outcome. If they choose something else they must be doing at least weakly better.

5. Consider the following sequential game (or extensive form game of complete information.)



(a) Find the best response at each decision node (or after every non-terminal history). *You may mark them above but explain your notation below.*

$$\begin{aligned} BR_1(I,O) &= O \\ BR_2(C,D) &= C \\ BR_2(A,B) &= B \\ BR_1(L,M,R) &= M \end{aligned}$$

(b) Find the subgame perfect equilibrium.
$((M,O),(B,C))$

(c) Using this game explain why it is important to write down the equilibrium strategies instead of the tactics or the outcome.

If I write the outcome $(2;1)$ *or the tactics* (M) *in equilibrium this doesn't verify that this is actually an equilibrium. If 1 plays I at their last decision node, then 2 will play D and then M is not a best response. Likewise if 2 plays A instead of B then M is not a best response. Thus I can not verify that people are always best responding if I just write down the outcome or tactics, and thus these descriptions do not specify an equilibrium.*

6. What is the definition of a (pure strategy) subgame perfect equilibrium. Be sure to define any technical terms you use in the definition. Notice your answer only has to cover the games that we have analyzed so far.

A subgame perfect equilibrium is a Nash equilibrium in every subgame.

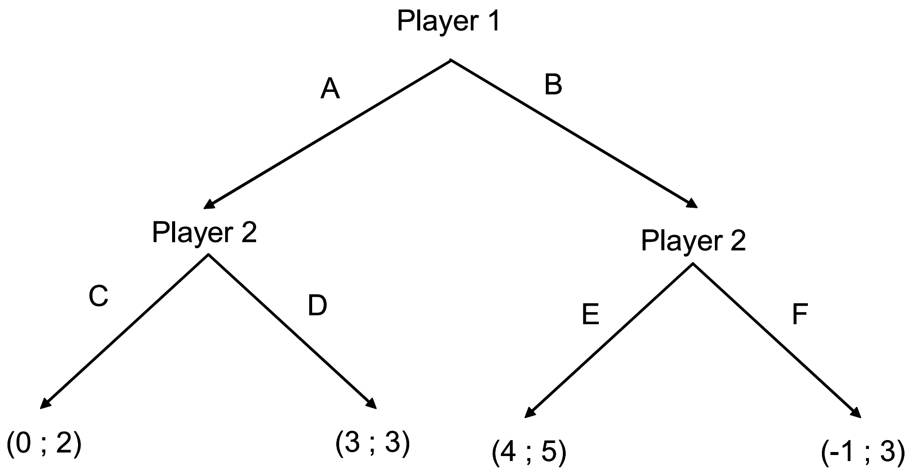
A subgame is a decision node where everyone knows that they are at that decision node and decision nodes which follow this one. In other words if you drew it as a separate game it would be a completely well-defined game.

A (pure strategy) Nash equilibrium is an action profile a^* *such that for every* i

$$a_i^* \in \arg\max_{a_i \in A_i} u_i\left(a_i, a_{-i}^*\right)$$

where A_i *is the set of strategies for player* i *in the appropriate subgame.*

7. Consider the following sequential game:



(a) Solve the game using backward induction, and write down the equilibrium *strategies* below.

D is the best response to A, E is the best response to B. Given these best responses B is the best response of player 1. Thus the equilibrium strategies are: $(B, (D, E))$

(b) Write down all of the strategies of both players.
Player 1: $\{A, B\}$
Player 2: $\{(C, E), (C, F), (D, E), (D, F)\}$

(c) Draw a normal form game that is strategically equivalent to the game above.

		Player 2			
		(C, E)	(C, F)	(D, E)	(D, F)
Player 1	A	$0; 2$	$0; 2^1$	$3; 3^2$	$3; 3^{12}$
	B	$4; 5^{12}$	$-1; 3$	$4; 5^{12}$	$-1; 3$

(d) Find the best responses and the Nash Equilibria of the normal form game.
The NE are $(B, (C, E))$ $(B, (D, E))$ $(A, (D, F))$

(e) Define an *empty threat.* Do empty threats always help the person who makes them?
An empty threat is an action that a person will not actually take if called upon to make a decision, but if the other person believes the threat then they will never be called on to make that decision. Empty threats may help someone, but not necessarily. In this game if player 2 threatens to play F if player 1 plays B then there will be a NE, but it will actually lower both parties payoffs.

(f) For the Nash Equilibria that are not Subgame Perfect equilibria explain how they depend on empty threats.
For $(A, (D, F))$ *it is clear, player 2's threat to play* F *if player 1 chooses* B *forces player 1 to choose* A.
For $(B, (C, E))$ *it is more subtle. Here player 2's threat is ineffective, but still this would not be an equilibrium because if player 1 chooses* A *player 2 will choose* D.

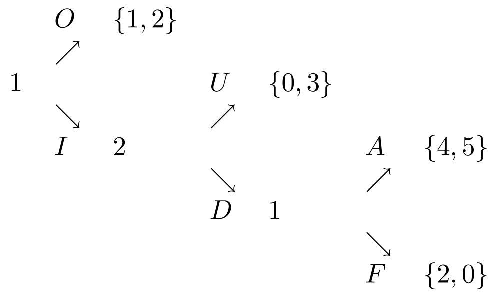
8. A strategy is *Subgame Perfect Dominated* if there is a subgame where it is strictly dominated.

Zermelo's Theorem tells us that as long as no two payoffs are the same for any person that in every Extensive Form Game of Perfect Information there is a unique Subgame Perfect Equilibrium.

Prove that there is also a unique strategy that survives iterated removal of subgame perfect dominated strategies.

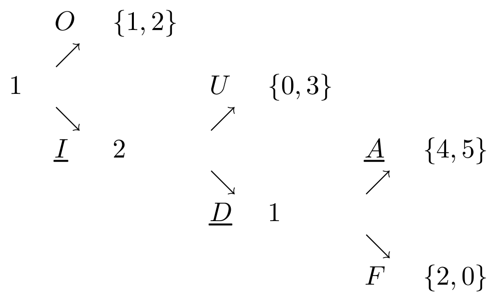
Assume there exists more than one strategy that survives iterated removal of subgame perfect dominated strategies. Let s_1 *and* s_2 *be two different such strategies and they are not equal to each other. Then* s_1 *and* s_2 *should differ at some node in the game tree. Apply backwards induction (Kuhn's Reduction algorithm) up to the nodes that are successors of the node that* s_1 *and* s_2 *differ from each other and the deviation should be the one closest to the terminal nodes. Now the remaining game tree also belongs to an* *extensive form game of perfect information and no two payoffs are the same for any person. Apply Zermelo's Theorem, and it tells that there is a unique subgame perfect equilibrium, which implies that $s_1$is strictly dominated by s_2 or vice versa, a contradiction.*

9. Consider the following sequential game (or extensive form game with perfect information.)



I will also describe this game using words. First 1 chooses between I and O, then if 1 chooses I then 2 chooses between U and D, then if 2 chooses D 1 chooses between A and F.

(a) Find the best responses of both players at every decision node and the Subgame perfect equilibrium.



I have underlined the best responses above, the SPE is $\{I, D, A\}$ *and the payoff is* $\{4, 5\}$

(b) List all of the strategies of both players.
$S_2 = \{U, D\}$, $S_1 = \{\{I, A\}, \{I, F\}, \{O, A\}, \{O, F\}\}$

(c) Draw a strategic form game that is equivalent to this sequential game.

	U	D
$\{I, A\}$	$0, 3$	$4, 5^{12}$
$\{I, F\}$	$0, 3$	$2, 0$
$\{O, A\}$	$1, 2^{12}$	$1, 2^{2}$
$\{O, F\}$	$1, 2^{12}$	$1, 2^{2}$

(d) Find the best responses of both players and all of the Nash equilibria of the game.

The best responses are marked by a 1 or 2 in the box above. The NE are $\{\{I, A\}, D\}$, $\{\{O, A\}, U\}$, $\{\{O, F\}, U\}$.

(e) Only one of these Nash equilibria is Subgame Perfect, explain what is wrong with the other Nash equilibria.

Solution 1 *i.* In general the problem is that the other Nash equilibria require player 1 or player 2 to take actions that they would never take if they were called on to act. Notice that in one of the NE player 2 is taking an action that is not sensible, and in the other player 1 is taking an action that is not sensible.
Notice that they are both kind of shooting themselves in the foot as well. In the SPE they get the Pareto efficient payoff, in the NE they are always doing much worse.

2 Chapter 6—Sequential Games, Illustrations.

1. An Extensive Form War of Attrition: Assume that there are two players, a and b, and let i be an arbitrary player, $i \in \{a, b\}$. In this game instead of deciding whether or not to fight for the next t periods in each period the two players decide whether or not to fight in the current period. Thus in period t a player's strategic choices are to fight (F_{it}) or Acquiesce (A_{it}). If a player choose to acquiesce in period t then the game ends and the other player wins the prize, otherwise you go to the next period. The payoffs are identical to the payoffs in the normal form game. Let t_i be the last period player i chooses to fight. Then:

$$u_i(t_i, t_j) = \begin{cases} 1 - c_i t_j & \text{if} \quad t_i > t_j \\ \frac{1}{2} - c_i t_i & \text{if} \quad t_i = t_j \\ -c_i t_i & \text{if} \quad t_i < t_j \end{cases}$$

where $c_i > 0$. We will only consider the game where $t \leq T$, where T is some fixed value.

(a) First consider $t = T$, and assume that both players have chosen to fight up to this point.

i. Draw the normal form game for this period.

	F_{bT}	A_{bT}
F_{aT}	$\frac{1}{2} - c_a T; \frac{1}{2} - c_b T$	$1 - c_a(T-1); -c_b(T-1)$
A_{aT}	$-c_a(T-1); 1 - c_b(T-1)$	$\frac{1}{2} - c_a(T-1); \frac{1}{2} - c_b(T-1)$

ii. Show that there is a $\bar{c}_T$ such that if $c_i < \bar{c}_T$ player i has a dominant strategy to choose F_{iT}.

$$\begin{aligned} F_{aT} &= BR_a(A_{bT}) \\ 1 - c_a(T-1) &> \frac{1}{2} - c_a(T-1) \\ 1 &> \frac{1}{2} \end{aligned}$$

$$\begin{aligned} F_{aT} &\in BR_a(F_{bT}) \\ \frac{1}{2} - c_a T &\geq -c_a(T-1) \\ \frac{1}{2} &\geq c_a \end{aligned}$$

$$\frac{1}{2} = \bar{c}_T$$

(b) Now consider $t = T-1$, and assume that $c_a < \bar{c}_T$ and $c_b < \bar{c}_T$.

i. Draw the normal form game for this period—be sure to consider what will happen next period if they choose (F_{aT-1}, F_{bT-1}).

	F_{bT}	A_{bT}
F_{aT}	$\frac{1}{2} - c_a T; \frac{1}{2} - c_b T$	$1 - c_a(T-2); -c_b(T-2)$
A_{aT}	$-c_a(T-2); 1 - c_b(T-2)$	$\frac{1}{2} - c_a(T-2); \frac{1}{2} - c_b(T-2)$

ii. Show that there is a $\bar{c}_{T-1}$ such that if $c_i < \bar{c}_{T-1}$ player i has a dominant strategy to choose F_{iT-1}.

$$\begin{aligned} F_{aT-1} &= BR_a(A_{bT-1}) \\ 1 - c_a(T-2) &> \frac{1}{2} - c_a(T-2) \\ 1 &> \frac{1}{2} \end{aligned}$$

$$\begin{aligned} F_{aT-1} &\in BR_a(F_{bT-1}) \\ \frac{1}{2} - c_a T &\geq -c_a(T-2) \\ \frac{1}{2} &\geq 2c_a \end{aligned}$$

$$\frac{1}{4} = \bar{c}_{T-1}$$

(c) For general t find a $\bar{c}_t$ such that if both players will fight from $t+1$ on player i has a dominant strategy to choose F_{it}.

$$u_a(F_{at}, F_{bt}) = \frac{1}{2} - c_a T \geq -c_a(t-1) = u_a(A_{at}, F_{bt})$$

$$\begin{aligned} \frac{1}{2} &\geq -c_a(t-1) - (-c_a T) \\ \frac{1}{2} &\geq c_a(T-t+1) \end{aligned}$$

$$\bar{c}_t = \frac{1}{2}\frac{1}{T-t+1}$$

(d) Show that the Subgame Perfect equilibria of this game still can be characterized as $t_a = 0$ and $t_b = k + \tau$ for $\tau \geq 0$ and a critical value of $k > 0$. Further show when $t_a = 0$ $t_b = T$ is not a Subgame Perfect Equilibrium.

If $t_a = 0$ *then it must be that* $1 - (t_b + 1)\, c_a \leq 0$*, thus let* $k = \frac{(1-c_a)}{c_a}$ *furthermore if* $t_a = 0$ *then it is optimal for* b *to choose* $F_{t,b}$ *for* $t \geq t_b$ *because his payoff is* 1 *as long as he chooses* $F_{1,b}$*.*

On the other hand if $0<t_a < t_b$ *then* a*'s payoff is* $-ct_a < 0$ *thus* $t_a = 0$ *is better.*

However if $t_b = T$ *then it may be that*

$$\begin{aligned} \frac{1}{2} - Tc_a &\geq 0 \\ \frac{1}{2c_a} &\geq T \end{aligned}$$

thus this is not an equilibrium.

If $t_a = t_b < T$ *then*

$$\begin{aligned} U_a(t_a, t_b) &= \frac{1}{2} - t_a c_a \leq 1 - (t_a + 1)\, c_a = U_a(t_a + 1, t_b) \\ \frac{1}{2} - t_a c_a &\leq 1 - (t_a + 1)\, c_a \\ c_a &\leq \frac{1}{2} \end{aligned}$$

Thus this could be an equilibrium if $c_a > \frac{1}{2}$ *and* $c_b > \frac{1}{2}$*, however we also need that deviating to* $t_a = 0$ *is not optimal:*

$$\begin{aligned} U_a(0, t_b) &= 0 \geq \frac{1}{2} - t_a c_a \\ t_a c_a &\geq \frac{1}{2} \end{aligned}$$

since $t_a \geq 1$ *both of these conditions can not be met.*

(e) Discuss how the equilibria of this model have the flavor of "the strong will win the prize." Or if $c_a < c_b$ then a will win. Be sure to discuss what you think a reasonable value for T is.

What we have found is that if T *is low enough and* $c_a < \frac{1}{2T} < c_b$ *then this is an equilibrium. Thus there is clearly a* T *such that this* *separation occurs as long as* $c_a \leq \frac{1}{2}$. *However* T *needs to be fixed in this model, so it will only occur by coincidence. It is obviously reasonable to assume that there is some finite* T *such that they will only fight for that many periods (or that many minutes) but in order to get a pure "the strong will win the prize" effect we need to be able to choose* T *properly, and we can't always be assured that we can do this. (Notice we also need* T *to be a continuous variable, but that is... not so important.)*

2. Someone wants to buy a used car which has value 40 with probability ρ_a, 12 with probability ρ_b and 6 with probability ρ_c ($\rho_a + \rho_b + \rho_c = 1$). The way the game proceeds is first the buyer makes an offer, p. Then nature (player 0) randomizes to determine which of the three types of cars the seller has, then the seller decides either to accept the offer or reject it. Let w be the value of the seller's car. The seller's payoff is p if he accepts the offer, and w if he rejects it. The buyers utility is $3w - p$ if his offer is accepted and zero if it is rejected. Assume that the seller will always accept the offer if he is indifferent between accepting it and rejecting it.

   (a) Show that $p \in \{40, 12, 6\}$, or the buyer will never make an offer that is not the value of some type of car. *Even if you can not show this you may assume it for the rest of the question.*

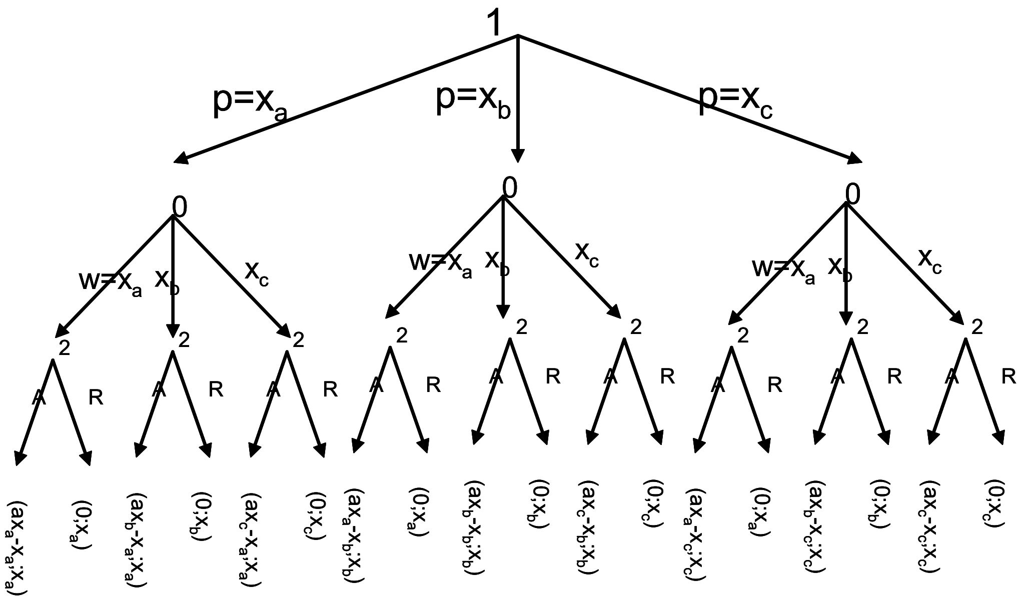
   Assume that $p > 6$, *then the sellers will always accept the offer and the buyers utility will be* $3E(w) - p$, *but the sellers will always accept* $p = 6$ *and this will give the buyer the utility* $3E(w) - 6$, *which is always higher.*

   Now assume $6 > p > 12$ *then the sellers utility will be* $(\rho_b + \rho_c) 3E(w|w \leq 12) - p$, *again the same sellers will accept* $p = 12$ *and the buyers utility will be higher.*

   Now assume $12 > p > 6$ *then the sellers utility will be* $\rho_c 3 * 6 - p$, *again the same sellers will accept* $p = 6$ *and the buyers utility will be higher.*

   Now consider $p < 6$, *in this case the buyers utility will be zero and since if* $p = 6$ *the buyers utility is* $3 * 6 - 6 > 0$ *the buyer will never want to do this.*

   (b) Assuming that $p \in \{40, 12, 6\}$, draw an extensive form game that describes this interaction. You do not need to calculate all the payoffs.



(c) Given that $\rho_a = \frac{1}{4}$, $\rho_b = \frac{1}{4}$ and $\rho_c = \frac{1}{2}$ find the buyer's expected utility from each of his actions.

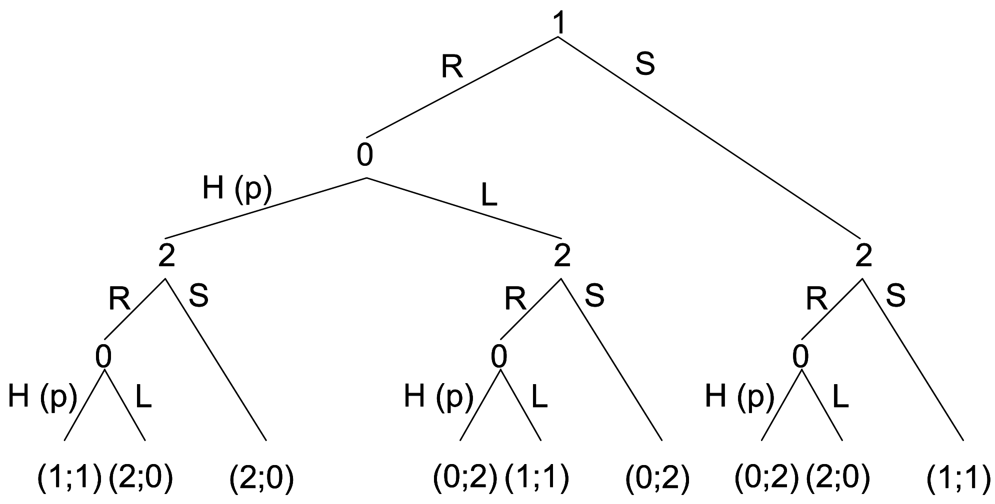
$$\begin{aligned}
Eu(40) &= 3E(w) - 40 \\
&= 3\left(\frac{1}{4} * 40 + \frac{1}{4} * 12 + \frac{1}{2} * 6\right) - 6 \\
&= 42 \\
Eu(12) &= \left(\frac{1}{4} + \frac{1}{2}\right)(3E(w|w \le 12) - 12) \\
&= \left(\frac{1}{4} + \frac{1}{2}\right)\left(3\left(\frac{\frac{1}{4} * 12 + \frac{1}{2} * 6}{\frac{1}{4} + \frac{1}{2}}\right) - 12\right) \\
&= 9 \\
Eu(6) &= \frac{1}{2}(3 * E(w|w \le 6) - 6) \\
&= \frac{1}{2}(3 * 6 - 6) \\
&= 6
\end{aligned}$$

(d) Find the optimal value for p in this game.
This will be whichever is highest of $Eu(40)$, $Eu(12)$, $Eu(6)$, *or*

$$p = \arg\max_{y \in \{40,12,6\}} Eu(y)$$

3. Consider the following sequential game with chance moves—or moves by nature. Nature is player 0, every time nature randomizes H occurs with probability $p = \frac{2}{3}$ and L occurs with probability $1-p$. **Assume throughout this question that** $p = \frac{2}{3}$.

(Story behind the game: In College American Football if there is a tie at the end of regulation time both teams get one chance to score. They can take the safe action (S) of kicking a fieldgoal—which will give them 3 points for sure—or they can take the riskier action (R) of trying to score a touchdown—which will give them 7 points if they succeed and 0 if they fail. Their probability of success is p. First team one makes their choice and both teams see the outcome, then team two makes their choice. The winner is the one with more points after both tries. A winner in this game gets two points, if the teams tie each gets one point. In College American football this process repeats if there is a tie, but for our analysis we can ignore this.)



(a) Find all the strategies of both players. You will be given one half point per strategy (rounded up). Be clear in your notation.

$S_1 = \{R, S\}$

$S_2 = \left\{ \begin{array}{l} (R(H), R(L), R(S)), (R(H), R(L), S(S)), (R(H), S(L), R(S)), (S(H), R(L), R(S \\ (S(H), S(L), S(S)), (S(H), S(L), R(S)), (S(H), R(L), S(S)), (R(H), S(L), S(S \end{array} \right.$

(b) Solve this game using backward induction. You may mark your answers on the game above but you will automatically loose one point if you do not explain your notation below.

Since $2 * \frac{2}{3} > 1$ $R(S)$. *Since if* L S *gets* 2 *with probability one* $S(L)$. $R(H)$ *because if* H *then* S *gets zero with probability one.*

$(R(H), S(L), R(S))$ *is the strategy of player 2. Now we need to* *construct the payoffs of player 1.*

$$\begin{aligned} U_1(S, R(S)) &= 2\left(1-\frac{2}{3}\right) \\ U_1(R, R(H), S(L)) &= \frac{2}{3}\left(2\left(1-\frac{2}{3}\right)+\frac{2}{3}\right) = \frac{2}{3}\left(2-\frac{2}{3}\right) = \frac{8}{9} \end{aligned}$$

to explain these payoffs, if $P1$ *chooses* S *then he will only win if player 2's risky strategy fails, this happens with probability* $1-\frac{2}{3}$*, in which case P1 will win outright. If P1 chooses* R *he will win only if he gets* H*, this will happen with probability* $\frac{2}{3}$*. Then* $P2$ *will choose* R *and P1 will win completely if that gamble fails* ($2\left(1-\frac{2}{3}\right)$) *and will tie otherwise, giving them a payoff of 1 with probability* $\frac{2}{3}$.

$$\begin{aligned} U_1(R, R(H), S(L)) &> U_1(S, R(S)) \\ \frac{2}{3}\left(2-\frac{2}{3}\right) &> 2\left(1-\frac{2}{3}\right) \\ \frac{2}{3} &> 0.585\,79 \end{aligned}$$

Since this is true for every variation of this question, P1 choose R.

(c) Write down the Subgame Perfect equilibrium strategies of both players. *Hint: When I say strategies I mean strategies, zero points will be awarded for incomplete strategies.*

$(R)\,(R(H), S(L), R(S))$

(d) Find the expected equilibrium payoffs of both players. Is player one's payoff higher than player two's or lower?

For player 1 it is above: $U_1(R, R(H), S(L)) = \frac{2}{3}\left(2-\frac{2}{3}\right)$ *for player 2 since this is a constant sum game* $U_1(R, R(H), S(L)) + U_2(R, R(H), S(L)) = 2$ $U_2(R, R(H), S(L)) = 2 - \frac{2}{3}\left(2-\frac{2}{3}\right)$

$$\begin{aligned} 2-\frac{2}{3}\left(2-\frac{2}{3}\right) &> \frac{2}{3}\left(2-\frac{2}{3}\right) \\ 2-2\frac{2}{3}\left(2-\frac{2}{3}\right) &> 0 \\ \left(\frac{2}{3}^2 - 2*\frac{2}{3}+1\right)\frac{2}{3} &> 0 \\ \left(1-\frac{2}{3}\right)^2 2 &> 0 \end{aligned}$$

which is always true, so P2 has the higher payoff.

(e) Now transform this game into a simultaneous action game where both players have two strategies, R and S. If they choose the strategy R they get the outcome H with probability $\frac{2}{3}$ and L with probability $1-\frac{2}{3}$.

i. Fill in the missing payoffs in the table below:

	R	S
R	$1;1^{12}$	$2*\frac{2}{3};2\left(1-\frac{2}{3}\right)^{1}$
S	$2\left(1-p\right);2*\frac{2}{3}^{2}$	$1;1$

ii. Find the best responses of both players. You may mark them in the table above but you will automatically loose one point if you do not explain your notation below.
Boxes that are a best response for player 1 have a 1 in the upper right hand corner, those for P2 have a 2 in the upper right hand corner.

iii. Find the unique Nash equilibrium in Pure Strategies.
(R,R)

(f) What does comparing the equilibrium payoffs in the simultaneous move game and the sequential game tell us about first mover's advantage? Explain your result. How does this explain the strategic decisions made in many sports?

In the simultaneous move game both parties get 1 in expectation, in the sequential game P1 gets $p\left(2-p\right)$ which is less than one. Thus obviously first movers advantage does not hold in games where there are random actions. The reason for this is that first mover's advantage only guarantees that you will get better payoffs than any pure strategy equilibrium, if you choose an action where nature randomizes you are playing a mixed strategy equilibrium. Thus we should not expect it to hold. This explains why in many sports (like downhill skiing) when a player is given a choice over going first or later they choose later. This way the random payoffs earlier athletes face are realized and the following player can make a better choice.

4. Consider a committee model where the set of options is $X=\{a,b,c,d,e\}$ and the preferences of the three people are:

1	2	3
a	e	c
b	b	d
c	c	e
d	d	b
e	a	a

An *agenda* is an ordering over the options, for example (b,e,c,a,d). The way options in an agenda are voted over is:

1. In the first round the first two options are voted over. The option that gets the majority of votes becomes the *status quo*.

t. In every following round the committee votes over the status quo versus the next option in the agenda. The option that gets the majority of votes becomes the new *status quo*.

The game ends when every option in the agenda has been considered. Assume that committee members always use weakly undominated strategies.

(a) For the agenda (a, b, c, d, e) find the outcome if:

i. Committee members vote *naively* (vote for their favorite option among the two options in front of them.)

1	2	3
a	e	c
b	b	d
c	c	e
d	d	b
e	a	a

But for naive voting it is quite simple. No one thinks about the future so one can start at the top of the agenda and go on from there. a is bottom ranked by $\{2, 3\}$ *so b wins. Then b beats c, then b beats d, then b looses to e. So e wins.*

ii. Committee members vote strategically.
Now we need to use backward induction, starting in the last round. I will use a table to answer this:

Round 4		Round 3		Round 2		Round 1	
Pairings	Winner	Pairings	Winner	Pairings	Winner	Pairing	Winner
(a, e)	e	(a, d)	d	(a, c)	c	(a, b)	Does not mat
(b, e)	e	(b, d)	d	(b, c)	c		
(c, e)	c	(c, d)	c				
(d, e)	d						

So the outcome is option c. Let me give an example so that you can understand my methodology. In Round 2 a vote for a is really a vote for d because that wins round 3 if a wins round 2, and then d will win in round 4. So the contest is between c and d. c Pareto dominates d, so it wins by majority.

(b) Define what it means to be *Pareto Efficient.* Find the Pareto efficient options for this committee.

An option is **Pareto Efficient** *if there is no feasible option that Pareto dominates it. An option Pareto dominates another if everyone prefers the first option. Another definition is that you can't make anyone happier without making someone else less happy.*

Notice that automatically all of the top ranked options are PE, so $\{a, c, e\}$ *get in by that criterion. Now for person 1 the only thing* *better than b is a, but no one else agrees, so b is PE* $\{a, b, c, e\}$. *Now for d the only thing better for 3 is c, but this is also better for 1 and 2, so d is not Pareto efficient.*

(c) Define what it means to be in the *Top Cycle*. Find the top cycle for this committee.

Something is in the **top cycle** *if it indirectly beats all other options.* x *indirectly beats* y *if there is a sequence of options* $\{z_k\}_{k=1}^{K}$ *and* x *beats* z_1, z_k *beats* z_{k+1} *and* z_K *beats* y. *I use the symbol* $x \blacktriangleright y$ *to say that* x *beats* y.

$a \blacktriangleright$	$b \blacktriangleright$	$c \blacktriangleright$	$d \blacktriangleright$	$e \blacktriangleright$
$\emptyset$	$\{a, c, d\}$	$\{a, d, e\}$	$\{a, e\}$	$\{a, b\}$

so we can form the cycle: $b \blacktriangleright c \blacktriangleright d \blacktriangleright e \blacktriangleright b$ *this is the largest such cycle and so it is the top cycle.*

(d) Find the set of options such that there is an agenda with this option as the equilibrium outcome for this committee.

This is the set of options that are PE and in the top cycle, thus it is $\{a, b, c, e\} \cap \{b, c, d, e\} = \{b, c, e\}$.

γ	p s.t $\{U, (R, O)\}$ is an equilibrium.
6	$\left[\frac{1}{7}, \frac{6}{7}\right]$
7	$\left[\frac{1}{8}, \frac{7}{8}\right]$
8	$\left[\frac{1}{9}, \frac{8}{9}\right]$
9	$\left[\frac{1}{10}, \frac{9}{10}\right]$

5. In the market for pocket calculators there are two firms. Firm 1 can either produce 7 or exit the industry (and never reenter). Firm 2 can either produce 3 or exit the industry (and never reenter). The costs of production are the same for both firms, $c_1(q) = 7q_1$, $c_2(q) = 7q_2$. This industry is in decline since cell phones, computers, PDA's and etcetera all do the same job as pocket calculators but do more interesting things as well. Thus the inverse demand curve in this industry is $P(t, Q_t) = \max\{25 - t - Q_t, 0\}$.

   Just to be clear, if firm 1 wants to produce in period 5 then they have to produce in every period before that date. If they choose to produce nothing in period 4 they have exited the industry and can never produce in this industry again (for example they must produce nothing in period 5). *Assume firms will produce if they are indifferent between producing and not producing.*

   (a) Define t_1 as the highest value of t such that $P(t, 7) \geq 7$, t_2 as the highest value of t such that $P(t, 3) \geq 7$, and t_b as the highest value of t such that $P(t, 10) \geq 7$. Find all three values. For $i \in \{1, 2\}$ what is the relationship between t_i and the last period firm i will produce?

$$\begin{aligned} 25 - t_1 - 7 &= 7 \\ t_1 &= 11 \\ 25 - t_2 - 3 &= 7 \\ 15 &= t_2 \\ 25 - t_b - 3 - 7 &= 7 \\ 8 &= t_b \end{aligned}$$

(b) Find the equilibrium strategy of both firms in $t > \max\{t_1, t_2, t_b\}$.
Since both firms will never make a positive profit form t onwards they will both shut down.

(c) Find the equilibrium strategy of both firms in $t \leq \max\{t_1, t_2, t_b\}$.
For $\min\{t_1, t_2\} < t \leq \max\{t_1, t_2, t_b\}$ *either both firms will be shut down or the firm with the lower capacity will produce.*
First consider $t_h = \min\{t_1, t_2\} = \min\{11, 15\} = 11$ *then the firm with a lower capacity can produce for four more periods. Assume they have produced up to this period and that the firm with higher capacity produces in this period. Then they will produce in this period if:*

$$\begin{aligned} [(25 - 11 - 7 - 3)\,3 - 7 * 3] + \Sigma_{t=12}^{15}\,[(25 - t - 3)\,3 - 7 * 3] &\geq 0 \\ [(25 - 11 - 7 - 3) - 7] + \Sigma_{t=12}^{15}\,[(25 - t - 3) - 7] &\geq 0 \\ [(25 - 11 - 7 - 3) - 7] + \Sigma_{t=12}^{15}\,[(25 - 11 + 11 - t - 3) - 7] &\geq 0 \end{aligned}$$

Note that $25 - 11 - 7 - 7 = 0$

$$-3 + \Sigma_{t=12}^{15}\,[(7 + 11 - t - 3)] \geq 0$$

Note that $7 = 3 + 4$

$$-3 + \Sigma_{t=12}^{15}\,(4 + 11 - t) \geq 0$$

and let $z = t - 11$

$$\begin{aligned} -3 + \sum_{z=1}^{4}(4 - z) &\geq 0 \\ 3 &\leq 6 \end{aligned}$$

and this is true for all the problems. Thus the low capacity firm will produce even if the high capacity firm will produce. Given the low capacity firm produces the high capacity firm will exit in 11.
Now consider $11 - 1$. *The profit the low capacity firm makes if it produces (assuming the other firm produces) is:*

$$[(25-11+1-7-3)\,3-7*3]+\Sigma_{t=11}^{15}\,[(25-t-3)\,3-7*3]\geq 0$$

and since $[(25-11+1-7-3)\,3-7*3]>[(25-11-7-3)\,3-7*3]$ *and*
$\Sigma_{t=11}^{15}\,[(25-t-3)\,3-7*3]\ >\ \Sigma_{t=12}^{15}\,[(25-t-3)\,3-7*3]$ *the low capacity firm will produce. By induction this logic continues until* 8. *Given after* 8 *there is only one future both firms will produce in period* 8 *and before.*

Thus the equilibrium is both firms produce to period 8, *after that only the low capacity firm produces until* 15, *and then neither firm produces.*

6. There are three committee members ($\{1,2,3\}$) and four alternatives ($\{a,b,c,d\}$). The preferences of the committee members are represented by the following table:

1	2	3
a	d	c
b	a	d
c	b	a
d	c	b

The alternatives are voted over in the following order. First committee members vote on b versus c, the winner being decided by majority voting. Then the winner is voted on against the outcome d, finally the winner is pitted against the outcome a.

(a) Prove that any outcome is a Subgame Perfect equilibrium of this game.

Assume that everyone believes that everyone else will always vote for $x\in\{a,b,c,d\}$ *when it is voted on, then whatever they vote for earlier will result in* x *winning, and if they vote for* $y\neq x$ *in any contest against* x *then it will not matter, thus* x *is a best response. Thus there is an equilibrium where* x *is selected.*

From this point on assume people use weakly undominated strategies.

(b) Find the outcome in the third round assuming each of the other alternatives $\{b,c,d\}$ wins the previous rounds.

If b wins, then at stage b versus a: agent 1 chooses a, agent 2 chooses a and agent 3 chooses a, thus a wins.

If c wins, then at stage c versus a: agent 1 chooses a, agent 2 chooses a and agent 3 chooses c, thus a wins.

If d wins, then at stage d versus a: agent 1 chooses a, agent 2 chooses d and agent 3 chooses d, thus d wins.

Hence {a,d} is the outcome set.

(c) Find the outcome in the second round assuming each of the other alternatives $\{b, c\}$ wins the previous round.

If b wins, a vote for b is a vote for a in the next round (since a wins) thus voters will vote for a versus d. A majority prefer d to a, thus d will win in the contest between b and d.

If c wins, then again this is a vote for a (via c) and a vote for d. Thus d wins.

Hence {d} is the outcome set.

(d) Find the outcome in the first round and the outcome of the game.

A vote for either option will result in d winning, thus the outcome does not matter.

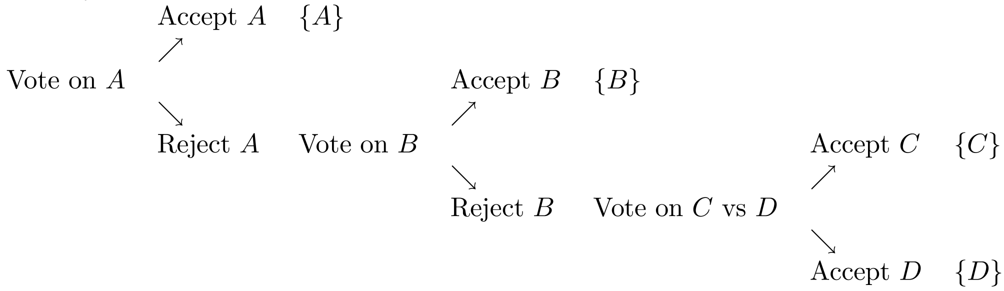
7. A committee consists of three members $\{1, 2, 3\}$. There are four options they are considering $\{A, B, C, D\}$. The preferences of the three members over the four outcomes are:

1	2	3
B	C	A
A	D	C
D	B	D
C	A	B

they have agreed that they should vote on options in the order of $\{A, B, C, D\}$ but are still arguing about the procedure to use. Assume that committee members always use weakly undominated strategies, or that when there vote will not matter they vote for the option that will lead to their preferred outcome.

(a) In this version they first vote on whether to accept option A or reject it and consider option B. They then vote on whether to accept option B or reject it and consider option C. Then they vote on whether to accept option C or accept option D. In each case the outcome is determined by the majority of voters.

i. Draw an extensive form game that represents this method. Please explain your notation.

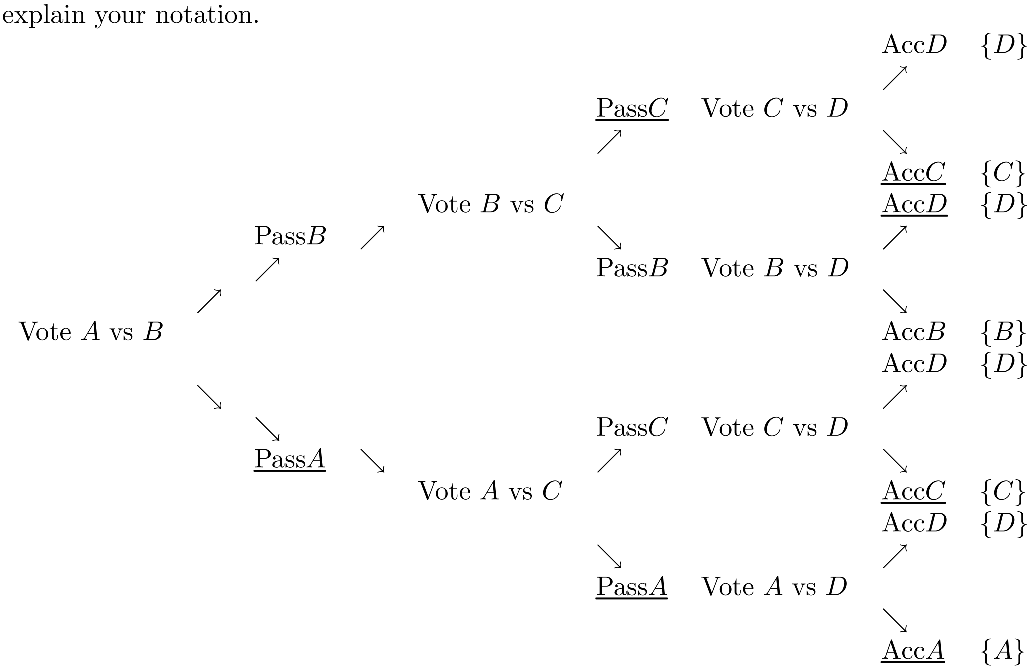


The places where that say Vote on X are simultaneous move games where everyone simultaneously votes either to accept or *reject X. The place where it says Vote on C vs D is a simultaneous move game where they vote for either C or D. The outcome is the item that is accepted, and is the same for all players.*

ii. Find the outcome of the voting at each stage and the outcome of this procedure.
In the last stage C beats D in the majority voting. In the next to last stage if they reject then they accept C, and since they prefer C to B they vote to reject B. In the first stage since two people prefer A to C they accept A, and that is the outcome.

(b) In this version they first vote on option A versus option B. Then the winner is matched against option C and then the winner of that contest is matched against option D. In each contest the outcome is determined by the majority of voters.

i. Draw an extensive form game that represents this method. Please explain your notation.



In this notation each Vote on X vs Y is a simultaneous action game where all committee members vote simultaneously. PassX means that the given item has passed that competition, AccX means that that item has been accepted as the outcome.

ii. Find the outcome of the voting in each pairing and the outcome of this procedure.
The outcome of each vote is underlined above, the outcome of the procedure is A.

3 Chapter 7

1. About Subgame Perfect Equilibria:

   (a) Define a Subgame

   A **sugbame** *is a the part of the game following some history h such that everyone knows that h has occurred. It must include all actions possible at h and all following actions.*

   (b) Define a Subgame Perfect Equilibrium, you may use the concept of Nash equilibrium without definition.

   A **Subgame Perfect Equilibrium** *is a Nash equilibrium in every subgame.*

2. Consider a game of Bank Runs. There are I people who can either deposit money in the bank (B) or their mattress (M). If they deposit money in their mattress they always get a return of 1. If $2 \leq K \leq I$ deposit their money in the bank then those who deposit money in the bank get a return of $1 + r$, where $r > 0$, otherwise they get zero.

   (a) Consider this as a simultaneous game, in other words all players choose between M and B at the same time.

   i. Find the best response for each player.

$$BR_i(S_{-i}) = \begin{cases} B & \text{if at least } K-1 \text{ other people are playing } B \\ M & \text{else} \end{cases}$$

   ii. Find the pure strategy Nash equilibria of this game. Prove that they are equilibria.

   There are two, obviously, one in which everyone plays B and one in which everyone plays M. To show these are the only equilibria assume at a random distribution we have $K-1$ players playing B. Then all players who currently aren't playing B will switch to B, this obviously holds in the reverse if there are strictly less than $K-1$. Notice that if there are exactly $K-1$ playing B then one of these could first switch to M, this is the only distribution where this is possible.

   (b) Now consider this as a sequential game, where first player 1 chooses between M and B, then player 2 chooses, and so on.

   i. Find the best responses of the last player to make a decision.

   the best response will be the same as in the simultaneous game.

   ii. Given those best responses, find the best responses of the next to last player to make a decision.

   Now for this person, the best response will be different. Assume there are $K-2$ people who have chosen to play B, if this player *then chooses to play B he knows the last player will play B. Thus his best response is:*

$$BR_i\left(S_{-i}\right) = \begin{cases} B & \text{if at least } K-2 \text{ previous players are playing } B \\ M & \text{else} \end{cases}$$

iii. Now find the unique Subgame Perfect equilibrium of this game. Prove your answer.
By induction it is obvious that for person $t \in (1, 2, 3, ..., I)$

$$BR_i\left(S_{-i}\right) = \begin{cases} B & \text{if at least } K-1-(I-t) \text{ previous players are playing } B \\ M & \text{else} \end{cases}$$

and since $I \leq K$, $K-1-I \leq 1$, thus for player 1, the only best response is B. Given this the same is true for 2, and so on.

iv. Find the other Nash equilibrium outcome of this game, specifying each player's strategy in this equilibrium.
Obviously the other Nash equilibrium is everyone playing B.

v. Now find the minimal size group that must be making an empty threat for the Nash equilibrium outcome to occur. Assuming only the minimal size group is making an empty threat, which players can *not* be making an empty threat? Use this to identify who must be in this minimal size group.
The minimal size group is of size $I-(K-1)$. This is because if there are K people using the subgame perfect strategy logically they can just ignore the other people, thus there must be only $K-1$ using the subgame perfect strategy, and thus $I-(K-1)$ must be using the other strategy.
This group can not include player 1, because his decision starts the process and he must use the subgame perfect strategy.
Now assume that player 1 choose M, can player 2 be using an empty threat strategy? Not if $K \leq I-1$, because otherwise he could choose B and start on the path to a subgame perfect equilibrium.
By induction if $K \leq I-t$, then player t must be using the subgame perfect strategy.
Thus the minimal size group must be made up of the last $I-(K-1)$ players to make a decision.

3. Consider the following T period extensive form game. In period $t < T$, if t is odd,

t.a Player 1 makes an offer $s_1 \in [0, 1]$

t.b Player 2 accepts or declines. If player 2 accepts then the game terminates with the payoffs: $u_1(s_1, A) = \delta^{t-1} s_1$ and $u_2(s_1, A) = \delta^{t-1}(1-s_1)$. If player 2 declines then the game goes to period $t+1$.

If t is even the only difference is that in $t.a$ player 2 makes the offer s_1 and player 1 accepts or declines in $t.b$.

If $t = T$ then the only difference is that in the case of a rejection $u_1(s_1, R) = u_2(s_1, R) = 0$.

Assume throughout that if a person is indifferent between accepting and rejecting that he will accept.

(a) Assume that $T = 1$.

i. Find the best response of player 2 in $1.b$.

$$u_2(s_1, R) = 0 \leq 1 - s_1 = u_2(s_1, A)$$

for all $s_1 \in [0, 1]$ thus

$$BR_2(s_1) = A$$

for all $s_1 \in [0, 1]$ *(notice I used the assumption to find this.)*

ii. Find the subgame perfect equilibrium value of s_1.

$$U_1(s_1, BR_2(s_1)) = s_1,\ s_1 \in [0, 1]$$

therefore $s_1^* = 1$.

(b) Assume that $T = 2$.

i. Find the subgame perfect best response of player 2 in $1.b$.
By symmetry in the last period $s_1^* = 0$ *thus they will accept anything that gives them a payoff greater than*

$$\begin{aligned} \delta(1 - 0) &\leq 1 - s_1 \\ s_1 &\leq 1 - \delta \end{aligned}$$

$$BR_2(s_1) = \begin{cases} A & s_1 \leq 1 - \delta \\ R & s_1 > 1 - \delta \end{cases}$$

ii. Find the subgame perfect equilibrium value of s_1 in $1.a$.

$$u_1(s_1, BR_2(s_1)) = \begin{cases} s_1 & s_1 \leq 1 - \delta \\ \delta * 0 & s_1 > 1 - \delta \end{cases}$$

thus $s_1^* = 1 - \delta$.

iii. Why didn't I ask you any questions about period 2?
Because of the symmetry of the problem I know what happens in period 2 just by looking at the answer to part a.

(c) Assume that $T = 3$

i. Find the subgame perfect best response of player 2 in 1.b.
By symmetry, if the game goes to period 2 then $s_1 = \delta$, *thus they will accept only if:*

$$\begin{aligned} \delta (1-\delta) &\leq 1 - s_1 \\ s_1 &\leq (1-\delta(1-\delta)) = (1-\delta(1-\delta))(1+\delta)\frac{1}{1+\delta} = \frac{1+\delta^3}{1+\delta} \end{aligned}$$

ii. Find the subgame perfect equilibrium value of s_1 in 1.a.
like above their best response is to set s_1 *to maximize their payoff,* $s_1 = \frac{1+\delta^3}{1+\delta}$

iii. Why didn't I ask you any questions about periods 2 and 3?
Again, the game from period 2 on is the game above where $T = 2$, *thus I can take the solution to that game as given.*

(d) Assume that T is odd, show that the subgame perfect equilibrium value of s_1 in 1.a is $s_1 = \frac{1}{1+\delta}\left(1+\delta^T\right)$.

Above we showed that this was true for $T = 1$ *and* $T = 3$, *thus we need to show it for general* T. *Notice that if this is true for* $T-2$ *then the equilibrium offer in period 3 will be* $s_1 = \frac{1}{1+\delta}\left(1+\delta^{T-2}\right)$. *This is because we are solving by backward induction, and the game from* $T = 3$ *to* $T = T$ *is exactly the same as the game from* $S = 1$ *to* $S = T - 2$.

Thus we need to show that given this the equilibrium offer in period 3 is $s_1 = \frac{1}{1+\delta}\left(1+\delta^{T-2}\right)$ *that the equilibrium offer in period 1 is* $s_1 = \frac{1}{1+\delta}\left(1+\delta^T\right)$.

In period 2 person 1 will accept anything for which:

$$s_1 \geq \delta\frac{1}{1+\delta}\left(1+\delta^{T-2}\right)$$

thus person 2 will offer $\delta\frac{1}{1+\delta}\left(1+\delta^{T-2}\right)$, *but then in period 1 person 2 will accept anything for which*

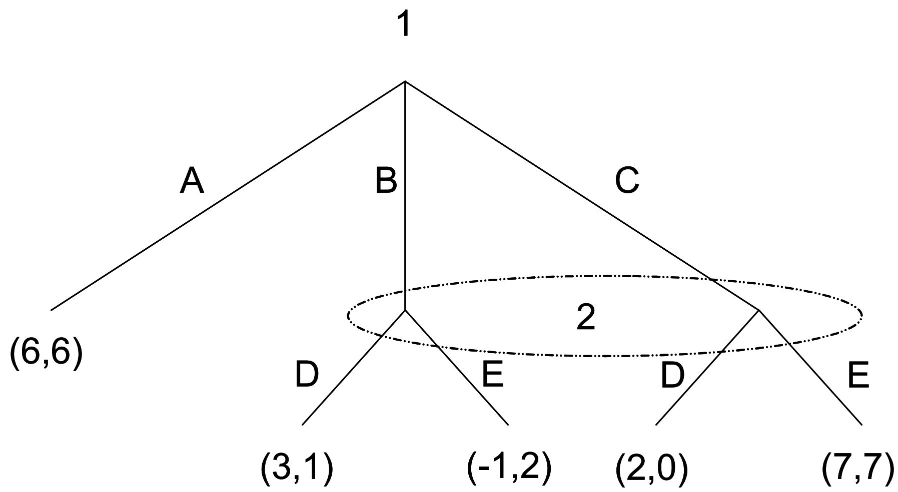
$$\begin{aligned} 1 - s_1 &\geq \delta\left(1-\delta\frac{1}{1+\delta}\left(1+\delta^{T-2}\right)\right) \\ 1-\delta\left(1-\delta\frac{1}{1+\delta}\left(1+\delta^{T-2}\right)\right) &\geq s_1 \end{aligned}$$

thus person 1 will offer:

$$\begin{aligned} s_1^* &= 1-\delta\left(1-\delta\frac{1}{1+\delta}\left(1+\delta^{T-2}\right)\right) \\ &= 1-\delta-\delta\left(-\delta\right)\frac{1}{1+\delta}\left(1+\delta^{T-2}\right) \\ &= 1-\delta+\frac{\delta^2\left(1+\delta^{T-2}\right)}{1+\delta} \\ &= \frac{\left(1-\delta\right)\left(1+\delta\right)}{1+\delta}+\frac{\delta^2\left(1+\delta^{T-2}\right)}{1+\delta} \\ &= \frac{1-\delta^2}{1+\delta}+\frac{\delta^2+\delta^T}{1+\delta} \\ &= \frac{1-\delta^2+\delta^2+\delta^T}{1+\delta} \\ &= \frac{1}{\delta+1}\left(\delta^T+1\right) \end{aligned}$$

4 Chapter 10

1. Consider the following general extensive form game.



   (a) Treating this as a game of complete information, find the Subgame Perfect equilibrium. You may solve this on the game above, but you will loose two points if you do not explain your notation below.
   $BR_2(B) = E, BR_2(C) = E, BR_1(\emptyset) = C$

   (b) Rewrite these strategies as strategies of the game as given. Are they a weak sequential equilibrium? If so be sure to carefully specify the beliefs, if not explain why not.

(C, E) and they are clearly a weak sequential equilibrium, beliefs must be $\Pr_2(C|B, C) = 1$ $\Pr_1(E|D, E) = 1$.

(c) Write down an equivalent normal form game for this extensive form game.

		Player 2	
		D	E
	A	$6;6^{12}$	$6;6^{2}$
Player 1	B	$3;1$	$-1;2^{2}$
	C	$2;0$	$^{1}7;7^{2}$

(d) Find the Subgame Perfect equilibria of this extensive form game.

Since there are no subgames this is just the Nash equilibria. The best responses are marked on the table above, with a 1(2) in the upper right hand corner if it is player 1's (2's) best response. Thus the Nash equilibria are (A, D) and (C, E)

(e) One of these Subgame Perfect equilibria is not a Weak Sequential Equilibrium. Explain the problem with this strategy.

The odd one is (A, D), this is not a Weak Sequential equilibrium because player 1 must believe that player 2 will play E if he plays anything other than A, so he is not best responding to his beliefs.

2. Consider a model of bank runs. In this question I am going to change notation from before, and make it a game of incomplete information, but otherwise it will be the same game as before. First of all, their are N players who all begin the game with one unit of money in the bank. Each player has the choice between withdrawing their unit of money and using it for purchases—W, or saving their money for the next period—S. If they withdraw their unit of money they get a return of $1 + \mu_i$, where μ_i has a known distribution $F(\cdot)$ over $[-1, (3)]$. To be clear only person i knows μ_i, for all other players it is only known to have the given distribution. If they choose to save their unit of money they get $1 + r$ if at least $2 \leq K \leq N$ people (including this person) choose S. **However we will only analyze the case where** $K = N \geq 2$. We assume that $r < (3)$. Throughout this analysis players will use a *cut-off* strategy and we will ignore the equilibrium where everyone chooses W unless it is the only equilibrium.

(a) For an arbitrary game, define a cutoff strategy and in this game explain what cutoff strategy players will use.

Let $\{X, Y\} \subseteq S_1$ and ε_1 be some information for player 1. Then a cutoff strategy is:

$$\sigma_1 = \begin{cases} X & \text{if} \quad \varepsilon_1 \geq \varepsilon_1^* \\ Y & \text{if} \quad \varepsilon_1 < \varepsilon_1^* \end{cases}$$

In this game if $\mu_i > r$ then the only best response is W thus the strategy is:

$$\sigma_i = \begin{cases} W & \text{if} \quad \mu_1 \geq \mu_1^* \\ S & \text{if} \quad \mu_1 < \mu_1^* \end{cases}$$

(b) First assume that $F(\mu) = \frac{1}{2}(\mu + 1)^{\frac{1}{2}}$, and that $N = 2$.

i. Consider the simultaneous game, where both players choose S or W at the same time. Find the equilibrium in symmetric strategies.

A. *Thus μ^* is defined as:*

$$\begin{aligned} 1 + \mu &= \frac{(\mu + 1)^{\frac{1}{2}}}{2}(1 + r) \\ (1 + \mu) &= (\mu + 1)\frac{(1 + r)^2}{4} \\ 1 + \mu &= \frac{(1 + r)^2}{4} \\ 1 + \mu &= \frac{(1 + r)^2}{4} \\ \mu^* &= \frac{(1 + r)^2}{4} - 1 \end{aligned}$$

ii. Consider the sequential game, where player 1 chooses S or W, then player 2 chooses S or W after seeing what player 1 does.

A. Prove that player 2 uses the Pareto Efficient strategy. I.e. S if Player 1 chooses S if $\mu_2 \leq r$, W otherwise.
This person has complete information about what player 1 is going to do, thus he will obviously choose S if $\mu_2 \leq r$ and W otherwise.

B. Find the equilibrium strategy for player 1.
Player 1 will choose μ_1^* such that:

$$\begin{aligned} 1 + \mu_1 &= \frac{(r + 1)^{\frac{1}{2}}}{2}(1 + r) \\ 1 + \mu_1 &= \frac{(r + 1)^{\frac{1}{2}+1}}{2} \\ \mu_1^* &= \frac{(r + 1)^{\frac{3}{2}}}{2} - 1 \end{aligned}$$

iii. Show that the probability that either player saves their money in the simultaneous model is strictly lower than the probability that either player saves his money in the sequential model. What does this tell you about the probability of bank runs in the two models? Explain this result.

$$\begin{aligned}
\mu^* &< \mu_1^* \\
\frac{(1+r)^2}{4} - 1 &< \frac{(r+1)^{\frac{3}{2}}}{2} - 1 \\
\frac{(1+r)^4}{16} &< \frac{(r+1)^3}{4} \\
(r+1) &< 4
\end{aligned}$$

which is true because $r \leq 1$. You also need to show that $\mu^* < r$, and this is also rather easy to do.

$$\begin{aligned}
\frac{(1+r)^2}{4} - 1 &< r \\
\frac{(1+r)^2}{4} &< 1 + r \\
(1+r) &< 4
\end{aligned}$$

again this is clear. This suggests that the sequential game the outcome will always be better than the simultaneous game. The intuition for this logic is straightforward. In the sequential game each person is better informed when they make their decision, thus their decision is less risky, thus they are more likely to use a strategy that is nearly Pareto Efficient.

(c) Now consider the general sequential game, where player $i \in \{1, 2, 3, ..., N\}$ choose S_i or W_i observing the choices of players $j < i$.

i. For this part of the question assume that $F(\cdot)$ is binary: with probability ρ $\mu_i = 3$ and with probability $1 - \rho$ $\mu_i = 0$.

A. Prove that player N will use the Pareto Efficient strategy. I.e. S_N if $\mu_N = 0$ and all other players have played S, W_N otherwise.

I gave another point for this? I hope you got these points, they're just easy.

B. Prove that there is an K^* such that if $i \geq N - K^*$ i will use the Pareto Efficient strategy. I.e. S_i if $\mu_i = 0$ and all previous players have played S, W_i otherwise.

In this part of the question, all we need to show is that

$$1 < (1-\rho)^{K^*} (1+r)$$

obviously if $\mu = 3$ he should and will choose not to cooperate, and all the following people will continue to cooperate. Obviously for small enough K^* he will use the Pareto efficient strategy. To be specific K^* is the unique K^* such that

$$\begin{aligned}
1 &\leq (1-\rho)^{K^*} (1+r) \\
1 &> (1-\rho)^{K^*+1} (1+r)
\end{aligned}$$

C. Prove that if $i < K^*$ player i will never choose S_i.
We just did this, for large enough K,

$$1 > (1-\rho)^{K+1}(1+r)$$

and thus this person will always choose W.

D. Find a explicit formula for K^*, and note that it is independent of N. Ignore the fact that it will usually not be an integer in this analysis. Comment on your result.
Ignoring the integer problem, K^* *is defined as:*

$$\begin{aligned} 1 &= (1-\rho)^{K^*}(1+r) \\ \frac{1}{(1+r)} &= (1-\rho)^{K^*} \\ -\ln(1+r) &= K^* \ln(1-\rho) \\ \frac{-\ln(1+r)}{\ln(1-\rho)} &= K^* \end{aligned}$$

and this result shows that if the population is too large the bank will always fail.

ii. Now assume that $F(\cdot)$ has an arbitrary distribution.

A. Prove that player N will use a Pareto Efficient strategy.
Man, I'm generous, I'm just handing out the easy points here.

B. Show that player $N-1$ will not use a Pareto Efficient strategy. I.e. the probability he will choose W_{N-1} is strictly positive even if $\mu_{N-1} < r$.
If this player invests he will get

$$F(r)(1+r)$$

if he holds his money he will get $(1+\mu)$. *Now assume* $\mu^*_{N-1} = r$, *but then we have*

$$\begin{aligned} 1 &> F(r) \\ (1+r) &> F(r)(1+r) \end{aligned}$$

so he strictly prefers to hold his money in this case, and $\mu^*_{N-1} < r$. *One can find an explicit formula in this case:*

$$1 + \mu^*_{N-1} = F(r)(1+r)$$

$$\mu^*_{N-1} = F(r)(1+r) - 1 \ .$$

which will always be strictly less than r *as long as* $F(r) < 1$.

C. Prove by induction that if $j < i$ then the probability j chooses W_j is either strictly higher than the probability i chooses W_i or equal to one.

To define μ_i^* *in this general setting let the probability that all following people invest be:* $\Pr(k > i \text{ invest})$ *Then*

$$1 + \mu_i^* = \Pr(k > i \text{ invest})(1 + r)$$

and we note that $\mu_i^* < r$ *thus* $F(\mu_i^*) \leq F(r) < 1$. *Then for person* $i - 1$ *we have:*

$$1+\mu_{i-1}^* = F(\mu_i^*)\Pr(k > i \text{ invest})(1 + r) < \Pr(k > i \text{ invest})(1 + r)$$

thus we must have $\mu_{i-1}^* < \mu_i^*$ *or* $\mu_i^* = -1$, *in which case we have* $\mu_{i-1}^* = \mu_i^* = -1$

D. Conclude that if N is large enough the only equilibrium is everyone playing W. Do you think this would also be true if $K < N$? Why or why not?
The conclusion is true because since $\mu_{i-1}^* < \mu_i^*$ *for large enough* T *we must have* $\mu_1^* = -1$. *To be precise we need to be more careful about the sequence here, but this is enough for half the credit. The other case, well I expect so but the proof is very hard, much beyond what I should expect of you in this class. An argument that "but banks are very large in reality" is not worth any credit here. A correct argument for it not being true would be "but in this case I don't need to worry about every person, just the "average" person, and for small enough K the average person should be willing to save their money."*
The argument for why it is true, as I said, requires more care. What we need to show is that for every $n \leq K$ *person* $\mu_i(n-1) < \mu_{i+1}(n)$. *To show this let the probability of a bank run given* $i+1$ *and* n *be:* $\Pr(BR|i+1, n)$ *then we know* $\mu_{i+1}(n)$ *is defined by:*

$$1 + \mu_{i+1}(n) = (1 - \Pr(BR|i + 1, n))(1 + r)$$

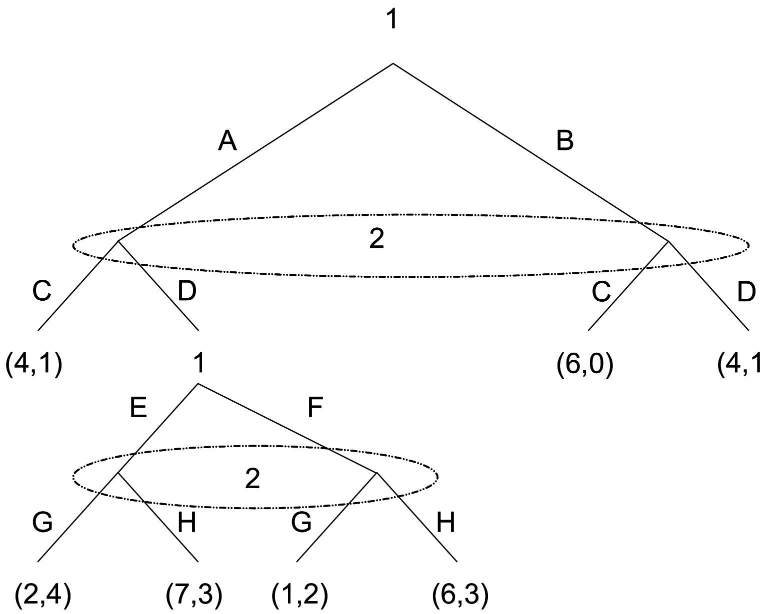
Now for person i *in state* $n - 1$

$$1+\mu_i(n-1) = F(\mu_{i+1}(n))(1 - \Pr(BR|i+1, n))(1 + r) < (1 - \Pr(BR|i+1, n))(1 +$$

thus $\mu_i(n-1) < \mu_{i+1}(n)$. *Now obviously* $\Pr(BR|i+1, n)$ *is strictly decreasing in* n, *thus* $\mu_i(n-2) < \mu_i(n-1)$ *but then, like before, for small enough* i $\mu_i(0) = -1$, *or this person will never put their money in the bank.*
This is weird, and I can't explain it intuitively. It seems that a bank should only get savings from a very small group of people, and this contradicts the massive size of most banks.
I think the reason this result is so counter-intuitive and yet true is because banks put a fraction of their savings into reserves for people who want to withdraw. Or $R = \rho N$, *thus*

$K = (1-\rho) N$ and I would guess that as long as $\rho \geq F(\mu^*)$ this result would be overturned. In a large enough population I know there are enough reserves for the average person to get their money.

3. Consider the following general extensive form game. Player 1 has the first decision, choosing between A and B.



(a) Write down all of the strategies of both players.
$S_1 = \{A, B\} \times \{E, F\} = \{(A, E), (A, F), (B, E), (B, F)\}$
$S_2 = \{C, D\} \times \{G, H\} = \{(C, G), (C, H), (D, G), (D, H)\}$

(b) Treating this as a game of complete information, find the Subgame Perfect equilibrium. You may solve this on the game above, but explain your notation below.
$BR_2(E) = G$
$BR_2(F) = H$
$BR_2(B) = D$
$BR_1(A, D) = F$
$BR_2(A) = D$
$BR_1(\emptyset) = A$

(c) Rewrite these strategies as strategies of the game as given. Are they a weak sequential equilibrium? If so be sure to carefully specify the beliefs, if not explain why not.

*$(A,F)(D,H)$ this is **not** a weak sequential equilibrium. To see this notice that if player 1 believes player 2 will play H then E gives a higher utility, and the best response to player 1 playing E is G.*

(d) Are there any other weak sequential equilibria? If so write them down and clearly specify the beliefs, if not prove this.

The first thing to check is if 2 plays G at his second information set. Then the best response of player 1 is to play E, given this player 2 will play D still, it is a dominant strategy. Then player 1 will choose B. Thus the strategies are: (B,E) (D,G) and some beliefs are $\beta_{22} = \Pr(F|E,F) = 0$, $\beta_{21} = \Pr(A|(A,B)) = 0$ the full set of beliefs for β_{22} are:

$$\begin{aligned} U_2(G,\beta_{22}) &= (1-\beta_{22})4 + \beta_{22}2 \geq (1-\beta_{22})3 + \beta_{22}3 = U_2(H,\beta_{22}) \\ \beta_{22} &\leq \frac{1}{2} \end{aligned}$$

(e) Are there any Nash equilibria outcomes that are not weak sequential equilibria? If so write them down and explain why they are Nash equilibria, if not prove this.

The surprising answer is no. To see this realize that the only strategies we have not considered are ones where player 2 plays C. But by construction no matter what happens in the subgame after (A,D) C is a dominated strategy. But player 2's first information set will be reached, thus he must play D. Thus we have considered all of his strategies, and all of them are weak sequential equilibria.

Of course there are other Nash equilibria, specifically if player 1 is planning on playing B we don't care what happens in the subgame (A,D), so we could have the strategy: (B,F) (D,G) This would require that $\beta_{22} = \Pr(F|E,F) = 0$ which is inconsistent with the actual strategies but not the equilibrium path.

4. About Weak Sequential Equilibrium:

(a) Define an *assessment.*

An assessment is a pair (σ,β) where σ_i is the strategy player i uses and β_i is his beliefs.

(b) Define the *sequential rationality* of an assessment.

An assessment is sequentially rational if for all $h \in H$ $\sigma_i(h)$ is optimal given $\beta_i(h)$

(c) Define the *consistency* of an assessment. You do not need to define Bayes' Rule.

An assessment is consistent if for all $h \in H$, $\beta_i(h)$ is consistent with $\sigma_{-i}(h)$, using Bayes' rule whenever possible (i.e. $\Pr(h|\sigma_{-i}) > 0$.

(d) Define a *Weak Sequential Equilibrium.*

A weak sequential equilibrium is an assessment that is sequentially rational and consistent.

(e) Why do we need to be so careful to specify beliefs properly in a weak sequential equilibrium?

Of course I will accept a wide range of answers for this question, but the correct answer is that in a game of incomplete information writing down strategies without beliefs is equivalent to writing down tactics in a game of complete information. If one writes down only tactics one has no way to test whether the strategies are optimal. Now since people sometimes don't know what has happened in the past if one doesn't write down beliefs then there is no way to know if their strategy is optimal.

5. A General Extensive Form War of Attrition: Assume that there are two players, a and b, and let i be an arbitrary player, $i \in \{a, b\}$. In this game instead of deciding whether or not to fight for the next t periods in each period the two players decide whether or not to fight in the current period. Thus in period t a player's strategic choices are to fight (F_{it}) or Acquiesce (A_{it}). If a player choose to acquiesce in period t then the game ends and the other player wins the prize, otherwise you go to the next period. The payoffs are identical to the payoffs in the normal form game. Let t_i be the last period player i chooses to fight. Then:

$$u_i(t_i, t_j) = \begin{cases} 1 - c_i t_j & \text{if} \quad t_i > t_j \\ \frac{1}{2} - c_i t_i & \text{if} \quad t_i = t_j \\ -c_i t_i & \text{if} \quad t_i < t_j \end{cases}$$

where $c_i > 0$. We will only consider the game where $t \leq T$, where T is some fixed value. Up to this point this is exactly the same game as you saw on the last exam. Let me remind you that we found on that exam that if $c_i < \tilde{c} = \frac{1}{2}\frac{1}{T}$ then i will fight to the end of the game.

Now we will transform this into a general extensive form game by assuming players have an unknown type. To be precise i knows $c_i \geq 0$ but all i knows about c_j is that it has a probability density function $g(c)$ and a cumulative density function $G(c)$ where $c \in [0, \infty)$. We assume that for any $0 < c < \infty$, $g(c) > 0$ and note that this implies $0 < G(c) < 1$ and that $\Pr(c_i < c) = \Pr(c_i \leq c) = G(c)$.

(a) Prove that there is a $\bar{c}_1$ such that if $c_i > \bar{c}_1$ then in any weak sequential equilibrium i chooses A_{i1}. *(Hint: Consider the best possible case for player i).*

The best case for player i is if $c_j \geq \tilde{c}$ then j chooses A_{j1}, however we know that if $c_j < \tilde{c}$ j will choose F_{i1}. Given these strategies, if player i chooses F_{i1} his utility will be:

$$(1 - c_i)(1 - G(\tilde{c})) - c_i G(\tilde{c}) = (1 - G(\tilde{c})) - c_i$$

if he chooses A_{i1} his utility will be:

$$\frac{1}{2}\left(1-G\left(\tilde{c}\right)\right)$$

and the latter is higher when:

$$\begin{aligned}\frac{1}{2}\left(1-G\left(\tilde{c}\right)\right) &\geq \left(1-G\left(\tilde{c}\right)\right)-c_i \\ c_i &\geq \left(1-G\left(\tilde{c}\right)\right)\frac{1}{2}\end{aligned}$$

thus $\bar{c}_1 = \left(1-G\left(\tilde{c}\right)\right)\frac{1}{2}$. To be precise there is no a-priori way of knowing that $\left(1-G\left(\tilde{c}\right)\right)\frac{1}{2} > \tilde{c}$, thus the precise answer is: $\bar{c}_1 = \max\left(\left(1-G\left(\tilde{c}\right)\right)\frac{1}{2}, \tilde{c}\right)$

(b) Given this fact, prove that there is a $\underline{c}_1$ such that if $c_i < \underline{c}_1$then in any weak sequential equilibrium i chooses F_{i1}.*(Hints: You may assume that in period 2 i choose A_{i2}. You should consider the worst possible case for player i, except that without loss of generality you can assume that if $c_j = \bar{c}_1$ player j chooses A_{j1}.)*

The worst case for player i is if $c_j < \bar{c}_1$ then j chooses F_{jt} for all t, however we know that if $c_j \geq \bar{c}_1$ j will choose A_{i1}. Given these strategies, if player i chooses F_{i1} his utility will be:

$$\left(1-c_i\right)G\left(\bar{c}_1\right)-c_i\left(1-G\left(\bar{c}_1\right)\right)=G\left(\bar{c}_1\right)-c_i$$

if he chooses A_{i1} his utility will be:

$$\frac{1}{2}G\left(\bar{c}_1\right)$$

and the latter is lower when:

$$\begin{aligned}G\left(\bar{c}_1\right)-c_i &\geq \frac{1}{2}G\left(\bar{c}_1\right) \\ G\left(\bar{c}_1\right)\frac{1}{2} &\geq c_i\end{aligned}$$

thus $\underline{c}_1 = G\left(\bar{c}_1\right)\frac{1}{2}$, and like before a precise answer is that $\underline{c}_1 = \max\left(G\left(\bar{c}_1\right)\frac{1}{2}, \tilde{c}\right)$

Just for the fun of it let me extend the argument after the first period. To do this we will assume that $\left(1-G\left(\tilde{c}\right)\right)\frac{1}{2} > \tilde{c}$, and we remind the reader that $\tilde{c}_t = \frac{1}{2}\frac{1}{T+t-1}$. Notice that in the second period we know that there are no players with $c_i \geq \bar{c}_1$, thus all probabilities have to be conditioned on $G\left(\bar{c}_1\right)$. Given this a player will play A_{i2} if

$$\begin{aligned}\frac{\left(G\left(\bar{c}_1\right)-G\left(\tilde{c}_2\right)\right)}{G\left(\bar{c}_1\right)}-c_i &\leq \frac{1}{2}\frac{\left(G\left(\bar{c}_1\right)-G\left(\tilde{c}_2\right)\right)}{G\left(\bar{c}_1\right)} \\ c_i &\geq \frac{1}{2}\frac{\left(G\left(\bar{c}_1\right)-G\left(\tilde{c}_2\right)\right)}{G\left(\bar{c}_1\right)}\end{aligned}$$

and thus $\bar{c}_2 = \max\left(\frac{1}{2}\frac{(G(\bar{c}_1)-G(\tilde{c}_2))}{G(\bar{c}_1)}, \tilde{c}_2\right) < \bar{c}_1$ *unless* $\bar{c}_2 = \tilde{c}_2$*. Given this conclusion a player will play* F_{i2} *if*

$$\begin{aligned}\frac{G(\bar{c}_2)}{G(\bar{c}_1)} - c_i &\geq \frac{1}{2}\frac{G(\bar{c}_2)}{G(\bar{c}_1)} \\ \frac{1}{2}\frac{G(\bar{c}_2)}{G(\bar{c}_1)} &\geq c_i\end{aligned}$$

and thus $\underline{c}_2 = \max\left(\frac{1}{2}\frac{G(\bar{c}_2)}{G(\bar{c}_1)}, \tilde{c}\right)$*. Notice in this case we can not be sure that* $\underline{c}_2 > \underline{c}_1$*. Thus by induction* $\bar{c}_t = \max\left(\frac{1}{2}\frac{(G(\bar{c}_{t-1})-G(\tilde{c}_t))}{G(\bar{c}_{t-1})}, \tilde{c}_t\right) < \bar{c}_{t-1}$ *(unless* $\bar{c}_t = \tilde{c}_t$*) and* $\underline{c}_t = \max\left(\frac{1}{2}\frac{G(\bar{c}_t)}{G(\bar{c}_{t-1})}, \tilde{c}_t\right)$*. Thus people with high enough costs will be constantly dropping out, until we are left with the hard core of players who will fight until the end.*

There is still room for a lot of different equilibria here, depending on the decision rules of players between $\underline{c}_t$ *and* $\bar{c}_t$*, thus we can only say that "if the strong are strong enough they will win."*

(c) We have now shown that in this generalized version of the war of attrition that the strong always win if they are strong enough. This matches empirical observation and is good.

Unfortunately in our current model there is frequently fighting, which does not match empirical observation.

Discuss how you could add a preliminary stage to the game that would remove the fighting in most cases. Specifically explain what the players have to do in this stage and what characteristics this interaction would have to take. Be careful to define all of the terms you use.

Would your method ever completely remove the fighting? Does your resulting model fit empirical observation? (To be concrete, consider the specific case where two bears come across a freshly killed deer at the same time.)

If we could include a signalling stage then this may decrease the probability of the fight.

A **signal** *is an action which would not be taken except for the informational content it provides—it would not be optimal except for this.*

A necessary characteristic of whatever signalling technology is used is that if someone is strong (c_i *is low) then it must be cheaper to send the signal than if that person is weak.*

It would not completely remove the fighting unless it was a perfect signalling technology. For example a physical exam of the two bears in question. But it would greatly reduce it. I argue that this is a correct empirical observation, bears do sometimes fight over the carcass—though usually only long enough to figure out who would win *a pitched battle. More often they rear back on their hind legs or do other things to signal that they are stronger. Pack animals usually establish a precedence ordering (establish the relative rank of their c_i's) and then occasionally have pitched battles when someone wants to change that order. Again, there are occasional fights—though again they are somewhat like a signalling stage in that they break off before someone dies.*

6. For a general extensive form game:

   (a) Define an *assessment.*

   An assessment is a strategy profile and beliefs about other player's strategies.

   (b) *Sequential Rationality:*

   An assessment is sequentially rational if the strategy is optimal given the beliefs at every decision node.

   (c) *Consistency* of an assessment. (You do not need to give a precise definition of Bayes Rule.)

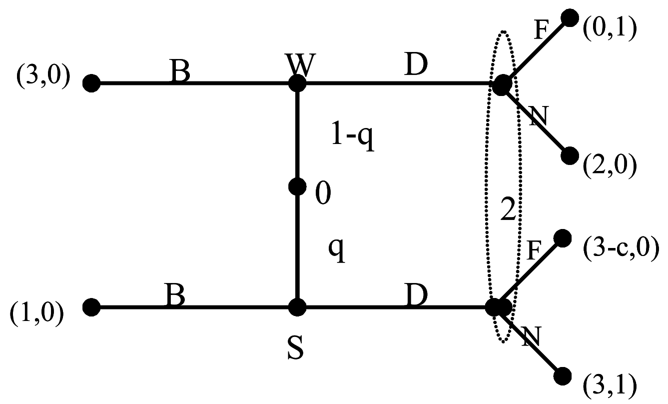
   Beliefs are consistent with other player's strategies if they are derived from those strategies using Bayes rule whenever possible.

   (d) *Weak Sequential Equilibrium:*

   An assessment is a weak sequential equilibrium if it is consistent and sequentially rational.

7. The Bully at the Door: You are eating in an otherwise empty restaurant when you hear someone yell at the top of their lungs: "I'm going to fight the next person who walks out this door." You look out the window and see a bully, not too strong looking, but a bully. Now you have two choices. You can sneak out the back (B) and leave the bully sitting there, or you can go out the front door (D) and face the bully. You may be one of two types: strong, S, with probability q or weak, W, with probability $1 - q$. While you know your type you know the bully will not be able to figure it out, but you know the bully knows q and you think he will not want to fight you if you are strong. Notice that the bully does have a choice between fighting (F) and not fighting (N) once you walk out the door.

The payoffs of this game are:



note that c is common knowledge.

(a) Write down all of the strategies of both players.
$S_1 = \{(B(W), B(S)), (B(W), D(S)), (D(W), B(S)), (D(W), D(S))\}$
$S_2 = \{F, N\}$

(b) Treating this as a game of complete information, solve the game using backward induction.
$BR_2(W, D) = F$, $BR_2(S, D) = N$, $BR(W) = B$, $BR(S) = D$.

(c) Write the strategy you just found as a strategy of the game as given, and solve for when it is a weak sequential equilibrium. Be sure to find off the equilibrium path beliefs, if there are any.
$(B(W), D(S))$, N. *The best response of player of type W is B always (he has a dominant strategy) and thus player 2 knows that if they observe D the player must be type S, thus she is best responding. Finally for type S (D, N) is the best payoff, so he is best responding. There is no off the equilibrium path, so no need to specify beliefs.*

(d) Which of the types of player one has a dominant strategy? Does the other one have dominant strategy sometimes? When?
Type W has a dominant strategy to play B, because $u_1(D, N, W) < u_1(B, W)$. I guess he's afraid that the bully might be crazy.
Type S has a dominant strategy to play D if $u_1(D, F, S) > u_1(B, S)$ or $3 - c > 1$, $c < 2$.

(e) Find another pure strategy weak sequential equilibrium. Be sure to check when this strategy is an equilibrium and find off the equilibrium path beliefs, if there are any.
Obviously the only strategy we haven't considered for player 2 is F, and we know that the best response of W to this strategy is D, so we need to know what the best response of S is. If it is D then clearly

this is not an equilibrium because player 2 does not want to fight type S. Thus the best response must be B. Thus the strategies must be $\{(B(W), B(S)), F\}$. In order for this to be a best response for type S we must have $c \geq 2$, since otherwise type S has a dominant strategy to play D. And now we have off the path beliefs. Let $\beta_2 = \Pr(S|D)$ then

$$\begin{aligned} u_2(F, \beta_2) &= \beta_2 * 0 + (1 - \beta_2) * 1 = 1 - \beta_2 \\ u_2(N, \beta_2) &= \beta_2 * 1 + (1 - \beta_2) * 0 = \beta_2 \end{aligned}$$

$$\begin{aligned} u_2(F, \beta_2) &\geq u_2(N, \beta_2) \\ 1 - \beta_2 &\geq \beta_2 \\ \beta_2 &\leq \frac{1}{2} \end{aligned}$$

Notice that in essence player 2 believes the deviating player to be weak, even though weak players have a dominant strategy to play B.

(f) Is there anything about the equilibria of this game that makes you doubt the bully's sanity?

My my, this is obviously just a fun question. But there are two things and I'll give the point for noticing either one.

i. *The bully never fights anyone in equilibrium. Obviously he is just wasting his time, and that is just plain silly.*

ii. *If the bully is planning on fighting, he believes that he will be fighting the weak even though the weak have a dominant strategy to avoid the fight.*

On the other hand, if the bully thinks you will believe his empty threat, then he may impress his friends by the fact that everyone sneaks out the back, in other words no one has the nerve to fight him. This will only work if $c \geq 2$.

8. Define the following terms, for simplicity I will give you the definition of an *assessment.*

Definition (σ, β) is an *assessment* if $\sigma = [\sigma_i]_{i=1}^{I}$ is a strategy and $\beta = [\beta_i]_{i=1}^{I}$ are beliefs about what other player will do in a game.

(a) *Sequential Rationality:*

σ is sequentially rational given β if for every sequence of play σ_i is a best response to β_i

(b) *Consistency* of an assessment:

β_i is consistent with σ_{-i} if it is obtained from σ_{-i} by Bayes rule whenever possible.

(c) *Weak Sequential Equilibrium:*
(σ, β) *is a weak sequential equilibrium if* σ_i *is sequentially rational with* β_i *and* β_i *is consistent with* σ_{-i}.

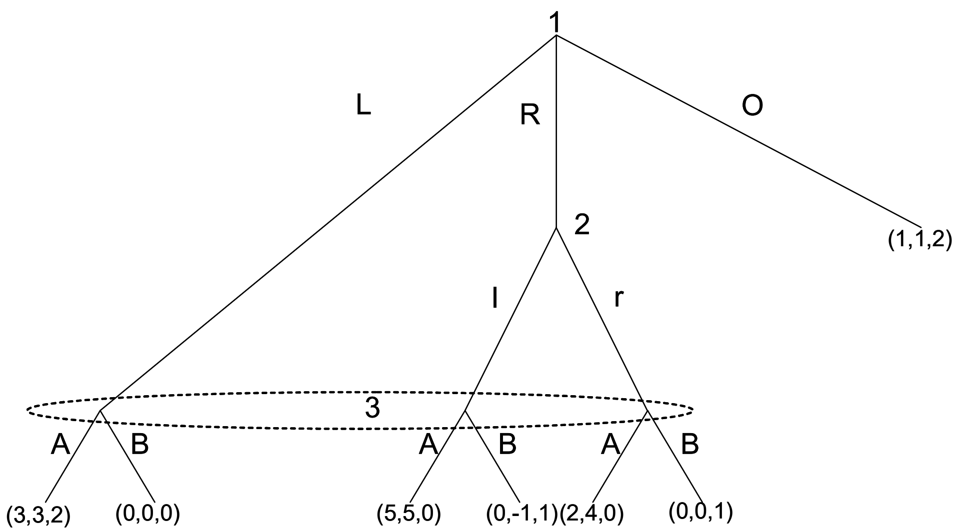
9. About *signals:*

   (a) Define a *signal*.
   A signal is an action that is taken both for the personal payoff and the information it provides to other parties. It may actually be worse than other actions in terms of personal payoffs, but the information it is sending is worth more than the cost in terms of direct payoffs.

   (b) We often say that education must be at least partially a signal. Explain why.
   The simplest reason is the extremely high difference in payoffs between almost having a degree and having a degree. If you have a undergraduate degree your first job will pay you a lot more than if you are about to complete a college degree.
   Alternatively you could use the fact that the training someone receives in their major is often different in different countries, but generally people do not care about that when making an offer.

   (c) Give two more examples of signals and explain why these things can be seen as signals. Two of the six points will be determined by the originality of your answers. I.e. if your signals were not things that we discussed in class.

10. Consider the following general extensive form game. Player 1 moves first and has three actions to choose between ($\{L, R, O\}$). Player 2 then chooses between l and r knowing what player 1 has chosen. Then player 3 chooses between A and B knowing only whether player 1 choose O or not.



(a) Write down all the strategies of all players in this game.
$S_1 = \{L, R, O\}$
$S_2 = \{l, r\}$
$S_3 = \{A, B\}$

(b) Treating this game as one of complete and perfect information, solve the game using backward induction. Then rewrite these strategies for the game as given, is this a weak sequential equilibrium of this game? A subgame perfect equilibrium? A Nash equilibrium?

$$\begin{aligned} BR_3(L) &= A \\ BR_3(R,l) &= BR_3(R,R) = B \\ BR_2(R) &= r \\ BR_1 &= L \end{aligned}$$

To write these strategies down as strategies of the extensive form game as given they are: L, r, A. *These are not weak sequential equilibrium because* $\Pr_1(A) = \Pr_2(A)$ *thus since player 1 believes* A *with probability 1 so should 2, and then* l *is the best response, in which case* R *is the best response for* 1.

However it is both a subgame perfect and Nash equilibrium, it is subgame perfect because the only subgame is the entire game. It is Nash because given the strategies of the others there is no way that one person deviating at a time can make their payoffs higher.

(c) Find a pure strategy weak sequential equilibrium for this game that doe not use the strategy you just found. Be careful to specify the beliefs of all players that make this a weak sequential equilibrium.

Since we just considered the case where 3 choose A, *there is only one case left to consider, where 3 chooses* B.

$$\begin{aligned} BR_2(B) &= r \\ BR_1(r,B) &= O \end{aligned}$$

The beliefs of players 1 and 2 are given by the strategies, in other words they think the probability someone plays the given strategy is one. All that is left is to specify player 3's beliefs. Let $\beta = \Pr_3(L)$ *then we need:*

$$\begin{aligned} Eu(A,\beta) &= \beta 2 + (1-\beta)\,0 \le \beta\,(0) + (1-\beta)\,1 = Eu(B,\beta) \\ \beta &\le \frac{1}{3} \end{aligned}$$

11. Consider a Spence signalling model. There are high and low productivity types. Education has no value in the workplace, but it costs less for high productivity workers to get an education. Thus firms believe that workers who get a more education are more likely to be high productivity.

To be precise the cost of e units of education for the high type is $c(E|H) = 2E$ and for the low type is $c(E|L) = 4e$. The productivity of the high types is 22 and the productivity of the low types is 10. The a-priori probability a worker is a high type is $\frac{2}{3}$, and after getting an education of length e a worker will earn their expected productivity. Let the firm's beliefs that a worker is a high type given e units of education be $\beta(e) = \Pr(H|e)$.

(a) What two types of equilibria will we find in this model? Explain each type.

Separating—a person's actions also reveals who they are.

Pooling—people of different types take the same action, thus not revealing who they are.

(b) If low types get no education and high types get 4, can this be an equilibrium or not? If it is specify a firm's beliefs that makes this an equilibrium, if it is not prove that it is not.

An example of beliefs are

$$\beta(e) = \begin{cases} 1 & \text{if} \quad e \geq 4 \\ 0 & \text{else} \end{cases}$$

this is an equilibrium if both types have the right incentives. Clearly low types have to not be willing to get 4.

$$\begin{aligned} 22 - 4 * m &\leq 10 \\ 22 - 10_l &\leq 4 * 4 \\ 3 &\leq 4 \end{aligned}$$

which is true in ever variation of the problem. High types have to be willing to get 4:

$$\begin{aligned} 22 - 2 * m &\geq 10 \\ 22 - 10 &\geq 2 * 4 \\ 6 &\geq 4 \end{aligned}$$

which is also always true.

(c) Find all equilibria where the two types of workers do not get the same level of education, be sure to prove your work and specify the beliefs.

specify such an equilibrium as a (h, l) *where if* $e \geq h$ $\beta(e) = 1$.

First of all, $l = 0$, *this is because the workers always have the outside option of getting no education, in this case the worst the firms can believe is that they are of type L. So this incentive constraint is:*

$$10 - 4 * l \geq 10 \rightarrow l = 0$$

Now the high types have to be willing to get h and the low types do not, so like above:

$$\begin{aligned} 22 - 4 * h &\leq 10 \\ 22 - 10 &\leq 4 * h \\ 3 &\leq h \end{aligned}$$

$$\begin{aligned} 22 - 2 * h &\geq 10 \\ 22 - 10 &\geq 2 * h \\ 6 &\geq h \end{aligned}$$

so $6 \geq h \geq 3$, *the beliefs are like before. For any* h *in this range:*

$$\beta(e) = \begin{cases} 1 & \text{if} \quad e \geq h \\ 0 & \text{else} \end{cases}$$

(d) If everyone gets 1, can this be an equilibrium or not? If it is specify a firm's beliefs that makes this an equilibrium, if it is not prove that it is not.

Now the only option workers have is getting 1 *and getting the average wage or not going to school at all and having firms believe they are the worst type.*

$$\beta(e) = \begin{cases} \frac{2}{3} & \text{if} \quad e \geq 1 \\ 0 & \text{else} \end{cases}$$

Notice that the beliefs they are type H must be λ *since this is the prior and everyone gets the education level* p. *So the low types will get this education if*

$$\begin{aligned} \frac{2}{3}22 + \left(1 - \frac{2}{3}\right) 10 - 4 * 1 &\geq 10 \\ \frac{2}{3}22 + \left(1 - \frac{2}{3}\right) 10 - 10 &\geq 4 * 1 \\ \frac{2}{3}(22 - 10) &\geq 4 * 1 \\ 2 &\geq 1 \end{aligned}$$

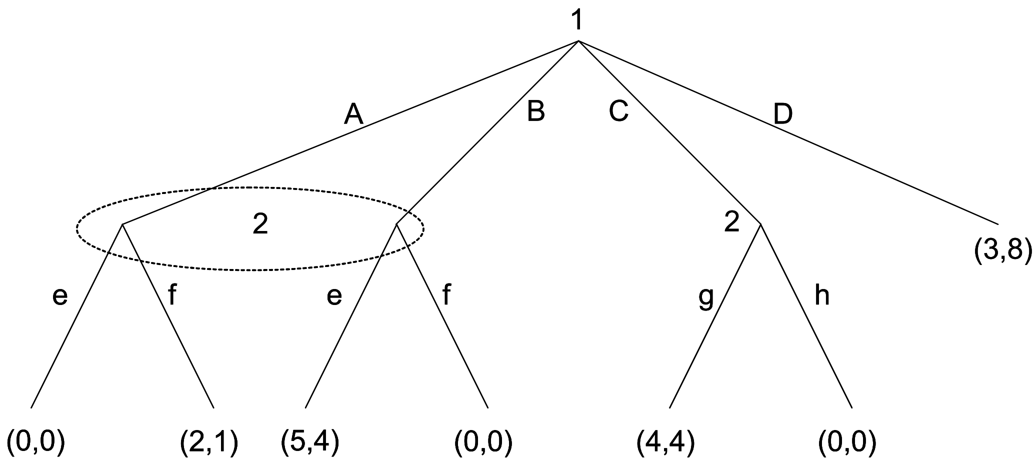
which is true for every variation. They high types will get it if:

$$\begin{aligned} \frac{2}{3}22 + \left(1 - \frac{2}{3}\right) 10 - 2 * 1 &\geq 10 \\ \frac{2}{3}22 + \left(1 - \frac{2}{3}\right) 10 - 10 &\geq 2 * 1 \\ \frac{2}{3}(22 - 10) &\geq 2 * 1 \\ 4 &\geq 1 \end{aligned}$$

(e) Find all equilibria where everyone gets the same level of education, be sure to prove your work and specify the beliefs.

For any p such that $2 \geq p$ *this proof was done in the last part of the question. In grading I won't require such detailed proof in the last part of the question..*

12. Consider the following extensive form game.



(a) Write down the strategies of both players.

1—$\{A, B, C, D\}$

2—$\{(e, g), (e, h), (f, g), (f, h)\}$

(b) Solve this game as if it was one of complete information using backward induction. *Only for this question should you treat this as a game of complete information.*

$BR_2(A) = f; BR_2(B) = e; BR_2(C) = g$

$BR_1(f, e, g) = B$

(c) Rewrite the strategies you just found as ones of the game as given and verify whether or not they are equilibrium strategies of this game.

The strategies are $\{B, (e, g)\}$ *and notice that all beliefs are specified since all information sets with imperfect information are reached.*

Since B is a best response to e and vice-versa these are in equilibrium, and since g is a BR to C this is an weak sequential equilibrium.

(d) Find all weak sequential equilibria of this game, be careful to write down the complete strategies and beliefs of both players.

One is specified above, so we don't have to rewrite it. Since C is a complete information history the only action possible after C is g. Thus there is only one other possibility: $(C, (f, g))$ *these strategies* *are best responses to each other only if f is a best response to either A or B being played. Let $\beta = \Pr_2(A|(A,B))$*

$$\begin{aligned} u_2(e,\beta) &= 0\beta + 4(1-\beta) = 4(1-\beta) \\ u_2(f,\beta) &= 1\beta + 0(1-\beta) = \beta \end{aligned}$$

$$\begin{aligned} u_2(e,\beta) &\leq u_2(f,\beta) \\ 4(1-\beta) &\leq \beta \\ \beta &\geq \frac{4}{5} \end{aligned}$$

any β in this range will be fine.

To do the rest of the analysis with a little more care:

(e,g)—$BR_1(e,g) = B$ above

(e,h)—$BR_2(C) = g$ not an equilibrium

(f,g)—$BR_1(e,g) = C$ analyzed more fully above.

(f,h)—$BR_2(C) = g$ not an equilibrium

(e) Find a Nash equilibrium of this game that is not a weak sequential equilibrium, and explain how this strategy relies on empty threats to be an equilibrium.

First of all notice that in this game there is no subgame, so we are merely looking for Nash equilibria. there two candidates.

In both of them the empty threat is to play h after c, in one of them this ends up having no effect because the equilibrium is the same as above. In the other one the outcome is (D), and this is only an equilibrium because the better outcome for player 1 (C,g) has been removed via an empty threat.

(e,h)—$BR_1(e,h) = B$

(f,h)—$BR_1(f,h) = D$ and these are both Nash equilibria, notice that the first one results in the same outcome as a w.s.e.

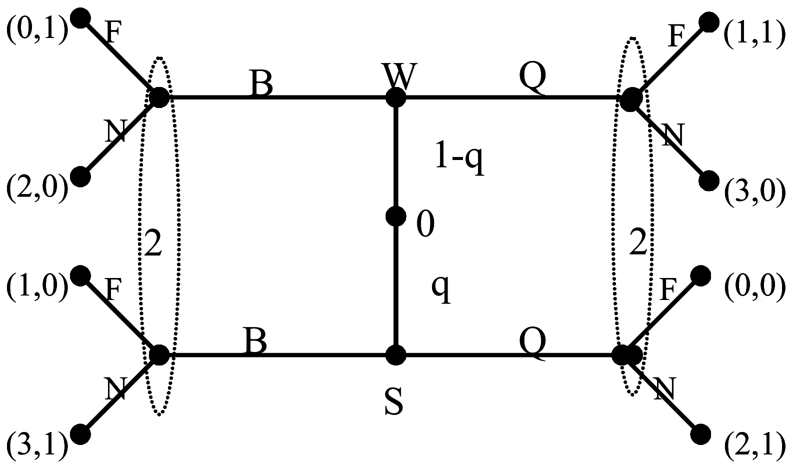
13. Give a precise definition of a weak sequential equilibrium, being sure to define any technical terms you use. *(You do not have to define "Bayes' rule.")*

A weak sequential equilibrium is a set of beliefs and strategies such that the strategy is a best response to the beliefs at every information set and the beliefs is derived from the players' strategies whenever possible.

The strategies must be sequentially rational (best response on every sequence of play) and the beliefs have to be consistent with the other player's strategies, specifically using Bayes rule, whenever possible.

14. Consider the following Beer/Quiche game. The way the action happens in this game is that first nature (player 0) determines whether player 1 is

weak (W) or strong (S). Player 1 is strong with probability $q \in \left[\frac{1}{2}, 1\right]$. Then player 1 decides whether to drink Beer (B) or eat Quiche (Q) for breakfast. (Notice that weak players prefer Quiche and strong players prefer Beer.) Then player 2 decides whether to Fight (F) player 1 or Not (N). Player 2 does not observe whether player 1 is strong or weak but does see what player 1 eats or drinks for breakfast. (Notice that player 2, the bully, only wants to fight a weak player 1, and that player 1 never likes getting in a fight.)



(a) Assuming player 2 can observe whether player 1 is weak or strong solve this game using backward induction. You can write your answer on the extensive form game but explain your notation below.

Note: From now on analyze the game using the information structure as given above. DO NOT treat it as a game of complete information.

$BR_2(W,Q) = BR_2(W,B) = F$, $BR_2(S,Q) = BR_2(S,B) = N$, $BR_W(F(B), F(Q)) = Q$, $BR_S(N(B), N(Q)) = B$

Note: From now on analyze the game using the information structure as given above. DO NOT treat it as a game of complete information.

(b) Rewrite the strategy you just found as a strategy in the game as given. Verify whether this is or is not an equilibrium strategy.

Player 2 must use the strategy $(N(B), F(Q))$*, and* $(Q(W), B(S))$ *is player 1's strategy. But* $u_1(W,B,N) = 2 > u_1(W,Q,F) = 1$*. So this is not an equilibrium.*

(c) Write down player 2's four strategies.

$(F(B), F(Q))$
$(N(B), F(Q))$
$(F(B), N(Q))$
$(N(B), N(Q))$

(d) For each of the four strategies of player 2 write down player 1's best response.

$BR_1(F(B), F(Q)) = (Q(W), B(S))$
$BR_1(N(B), F(Q)) = (B(W), B(S))$ because $u_1(W, B, N) = 2 > u_1(W, Q, F) = 1$
$BR_1(F(B), N(Q)) = (Q(W), Q(S))$ because $u_1(S, Q, N) = 2 > u_1(S, B, F) = 1$
$BR_1(N(B), N(Q)) = (Q(W), B(S))$

(e) For each of the best responses of player 1 (to the four strategies of player 2) write down player 2's best responses. If necessary give conditions when these are best responses.

$BR_2(BR_1(F(B), F(Q))) = BR_2(BR_1(N(B), N(Q))) = BR_2(Q(W), B(S)) = (F(Q), N($
$BR_2(BR_1(N(B), F(Q))) = BR_2(B(W), B(S)) = \{(N(B), F(Q)), (N(B), N(Q))\}$ when
$BR_2(BR_1(F(B), N(Q))) = BR_2(Q(W), Q(S)) = \{(F(B), N(Q)), (N(B), N(Q))\}$ when

To establish the last two claims, the beliefs of player 2 about player 1 must be that they are S with probability q thus:

$$
\begin{aligned}
u_2(F, q) &= 1 - q \\
u_2(N, q) &= q
\end{aligned}
$$

$$
\begin{aligned}
u_2(N, q) &\geq u_2(F, q) \\
q &\geq 1 - q \\
q &\geq \frac{1}{2}
\end{aligned}
$$

(f) What are the equilibria of this game? Write down the beliefs of player 2 in each of these equilibria (both on and off the equilibrium path.)

$(N(B), F(Q)), (B(W), B(S))$
$(F(B), N(Q)), (Q(W), Q(S))$

the on path beliefs are given by Bayes rule $\Pr(S|X) = q$ *where* X *depends on the equilibrium, it is* B *in the first one and* Q *in the second one. The off the path beliefs have to be that* $\Pr(S|-X) \leq \frac{1}{2}$*, for example* $\Pr(S|-X) = 0$ *would always work. (Where* $-X = \{B, Q\} \setminus X$*).*

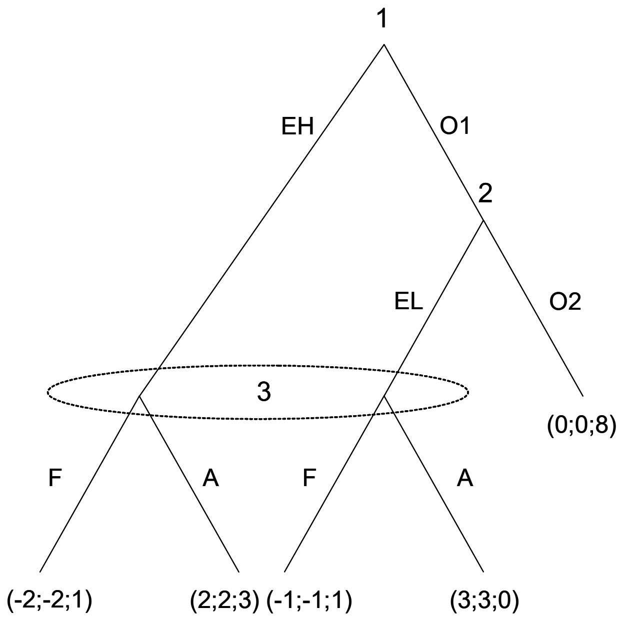
(g) What is a *signal*? Is there signalling going on in the equilibria you found for this game?

A signal is an action which is taken to inform others about some unobservable characteristic. The direct (marginal) benefit of the action is usually lower than the direct (marginal) cost, but the action is taken because of its informational content.

Whether or not there is signalling here is something that you probably could feel like debating, but there is indeed signalling going on in both of these equilibria. For example say that everyone is drinking beer. Then by drinking beer a weak person is indicating that they value not fighting more than their favorite breakfast. It is a signal, though *in this case what they are doing is signalling they don't want to be differentiated from the strong.*

If you accept that argument then you should also accept that when everyone is eating quiche the strong want to signal that they would rather not fight than have their favorite breakfast. Even though out of equilibrium the bully would not want to fight with them, in this equilibrium eating quiche is the way to signal you don't want to fight.

15. Consider the following entry game. Players 1 and 2 are in a partnership where they split the profits from entering an industry. Player 1 can choose to enter with high costs (EH) or stay out of the market (O1). If player 1 chooses to stay out then player 2 can over rule this decision and enter as a low cost firm (EL) or they can agree and the partnership will not enter. Player 3 (the incumbent firm) can choose to Fight (F) or not (A,accommodate). Player 3 does not know if the entering partnership is high cost (EH) or low cost (EL).



(a) Solve this game as one of complete information. Then rewrite the strategy you found as an equilibrium strategy in the game as given. Is this strategy a Nash Equilibrium? Is it a Subgame Perfect Equilibrium? Is it a Weak Sequential equilibrium?

3—$F(EL), A(EH)$, 2—$O2$, 1—EH. To rewrite this as a strategy of the game as given we just assume that player 3 chooses F.

It is a Nash equilibrium because player 1 and 3's actions are strict best responses, player 2 never makes a choice in equilibrium thus his *action is a best response. It is automatically a subgame perfect since there are no subgames.*

It is not a Weak Sequential equilibrium because given player 3's strategy if called on to make a choice player 2 should choose EL.

(b) For each action of player 3 solve the game using backward induction. Are any of the strategies you found Weak Sequential Equilibrium strategies?

3—F 2—O2 1—O1 since 3 does not make a choice all we need is for him to believe that player 2 choose EL with high enough probability (conditional on him having to make a choice) thus this is a WSE.

3—A 2—EL 1—O1 but A is not the best response to EL, thus this is not a WSE.

(c) For each weak sequential equilibrium strategy you found in the last part of the question find the beliefs of all players that make these strategies equilibria.

Most of these beliefs are trivial, let $\beta_i(X)$ be the probability of X for player i.

$$\begin{aligned}\beta_1(F|FA) &= \beta_2(F|FA) = 1\\ \beta_1(O2|O2,EL) &= \beta_3(O2|O2,EL) = 1\\ \beta_2(O1|O1,EH) &= \beta_3(O1|O1,EH) = 1\end{aligned}$$

the only difficult one is the beliefs of player 3, $\beta_3(EH|EH,EA)$ I will solve for the full range of beliefs, even though you only needed to specify a belief.

$$\begin{aligned}U_3(F,\beta_3) &= 1\\ U_3(A,\beta_3) &= \beta_3 3 + (1-\beta_3)0 = 3\beta_3\\ U_3(A,\beta_3) &= 3\beta_3 \geq 1 = U_3(F,\beta_3)\\ \beta_3 &\geq \frac{1}{3}\end{aligned}$$

so for example $\beta_3 = 1$ would be an acceptable answer.

16. A monopolist has just created a new product. The value of the product is unknown to the consumers, it will be $v_h = 15$ with probability $q = \frac{2}{3}$ and $v_l = 6$ the rest of the time. In order to reassure consumer's of the quality of the good the can offer a warranty (w) of either $0, 1, 2, 3, 4, 5$, or 6 years (the warranty must be in one year increments.) If the good has a high value then a warranty costs $c_h = 2$ per year to the firm, if it is of low value then it costs $c_l = 4$ per year.

    In equilibrium the price offered for the good must be equal to the expected value of the good to a consumer. Thus an equilibrium can be described as a p_h offered if the warranty is w_h, a p_l offered if the warranty is w_l and a price $\tilde{p}$ that is offered if the warranty is any other length. If the good has a high value the firm will offer w_h, and if the good has a low value the firm will offer w_l.

(a) Write down the constraints on the firm that the warranty and prices must satisfy in any equilibrium.

$$\begin{aligned} p_h - c_h w_h &\geq p_l - c_h w_l \\ p_h - c_h w_h &\geq \tilde{p} \\ p_h - c_l w_h &\leq p_l - c_l w_l \\ p_l - c_l w_l &\geq \tilde{p} \end{aligned}$$

(b) Explain why if we want to find all of the equilibria then we can assume that $\tilde{p} = v_l$.

Because the price must be equal to the expected value of the good, and the lowest expected value will be v_l. Since this is the worst alternative it will create the largest set of equilibria.

(c) Find all of the equilibria where $w_h = w_l$. What kind of equilibria are these?

If $w_h = w_l$ then these are pooling equilibria, no one is signalling. This means that $p_h = p_l = qv_h + (1-q)\,v_l = \frac{2}{3}(15) + \frac{1}{3}(6) = 12$. In order for these to be equilibria we need:

$$\begin{aligned} qv_h + (1-q)\,v_l - c_h w^* &\geq v_l \\ qv_h + (1-q)\,v_l - c_l w^* &\geq v_l \end{aligned}$$

obviously only the second one needs to be checked.

$$\begin{aligned} 12 - 4w^* &\geq 6 \\ 6 &\geq 4w^* \\ \frac{3}{2} &\geq w^* \end{aligned}$$

so the warranty can be zero or one year.

(d) Find all of the equilibria where $w_h \neq w_l$. What kind of equilibria are these?

These are separating equilibria, thus $p_h = v_h = 15$ and $p_l = v_l = 6$ the inequalities are:

$$\begin{aligned} p_h - c_h w_h &\geq p_l - c_h w_l \\ 15 - 2w_h &\geq 6 - 2w_l \\ 15 - 2w_h &\geq 6 \\ 15 - 4w_h &\leq 6 - 4w_l \\ 6 - 4w_l &\geq 6 \end{aligned}$$

from the last one we can see that $w_l = 0$ *leaving us with:*

$$\begin{aligned} 15 - 2w_h &\geq 6 \\ 15 - 2w_h &\geq 6 \\ 15 - 4w_h &\leq 6 \end{aligned}$$

$$\begin{aligned} 15 - 2w_h &\geq 6 \\ 9 &\geq 2w_h \\ 4.5 &\geq w_h \\ 15 - 4w_h &\leq 6 \\ 9 &\leq 4w_h \\ 2.25 &\leq w_h \end{aligned}$$

so the warranty can be three or four years.

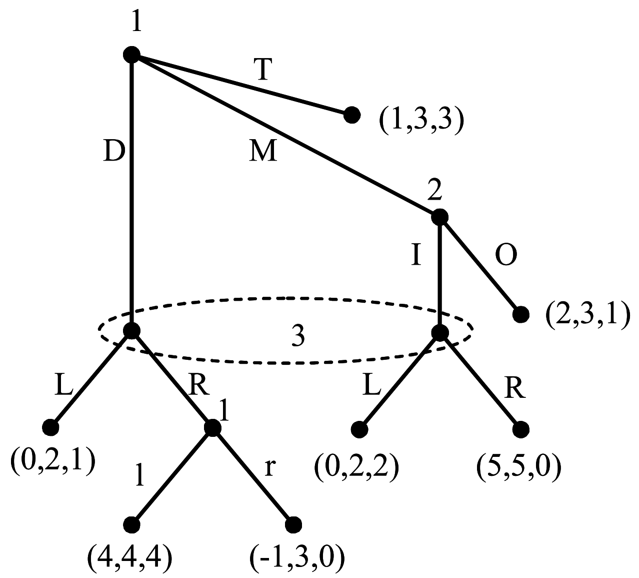
17. What is a Signal? Give two examples of signals in the real world.

    Solution 2 A Signal is an action that is taken in order to provide information to other people. The physical (marginal) cost of this action must be greater than the physical (marginal) benefit.

    There are many examples of signals in the real world. Some simple ones are:

    - Education—employers generally do not pay much attention to the content or even major of College graduates, thus the education you receive here has a partial signalling quality.
    - Owners paying for used cars to be inspected—clearly if the car will fail the inspection then the owner will not take it.
    - Warranties—if someone is selling low quality goods they will not want to offer warranties, thus only the trustworthy offer warranties.
    - Advertising—a lot of advertising neither provides information nor "good mood" it is merely conspicuous expenditure. This can be a signal.

18. Consider the following extensive form game:



(a) List all the strategies of the players.

1—$\{(T,l),(T,r),(M,l),(M,r),(D,l),(D,r)\}$ *2*—$\{I,O\}$ *3*—$\{L,R\}$

(b) Find all of the proper subgames, and find all Nash equilibria of these subgames. (A *proper* subgame is a subgame that is not the game itself).

There is only one subgame, where player 1 chooses between l and r, the equilibrium in this subgame is l.

(c) Treating this game as one of complete information.

i. Find the Backwards Induction equilibrium.

$1\,(R) - l, 3\,(D) - R, 3\,(I) - L, 2 - O, 1 - D$

ii. Write this as a strategy in the game above, I.e. when player 3 does not know if 2 played I or 1 played D.

$1 - D, 2 - O, 3 - R, 1 - l$

iii. Is the strategy you found in the last part a Nash equilibrium? A Subgame Perfect Equilibrium? A Weak Sequential Equilibrium? You must justify your answer for each part.

Solution 3 Nash Equilibrium: 2 never makes a decision, so O is a best response. Thus 3 knows that 1 must have chosen D, and she is best responding. Then clearly 1 is best responding to the other parties strategies.

Subgame Perfect Equilibrium: Yes, there is only one subgame and this equilibrium uses that strategy in the subgame. Since it is a NE of the rest of the game, it must be a SPE.

Weak Sequential Equilibrium: 2's beliefs about what 3 does must place probability 1 on R, in which case 2's best response is I. Given this 1 is no longer best responding, and the equilibrium unravels.

(d) Find all of the Weak Sequential Equilibria of this game. Be careful to specify the complete range of beliefs for player 3 that makes this an equilibrium.

Solution 4 Player 3 has two actions, L and R. L is the best response to I and R is the best response to D.

If 3 plays R then 2 will play I, then player 1 will play M, but then R is not a best response, so this can not be an equilibrium.

If 3 plays L then 2 will play O, and 1 will play M, Thus this is a candidate for a WSE. To precisely define it we must specify 3's beliefs, Let $\Pr(D|\{D,I\}) = \beta$

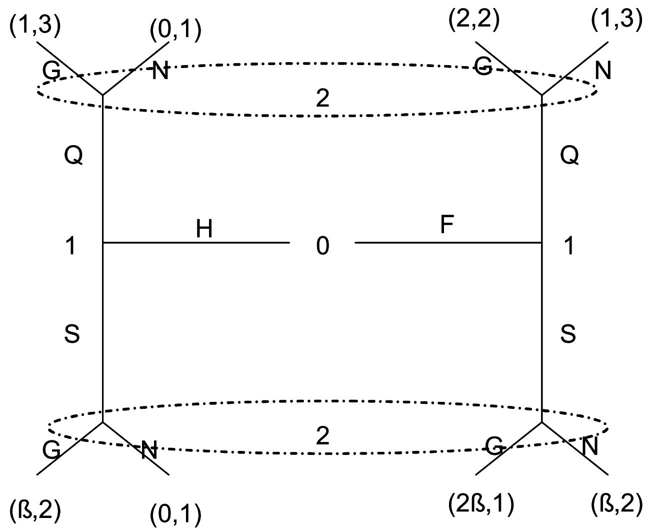
$$\begin{aligned} U_3(L,\beta) &= \beta + 2(1-\beta) \geq 4\beta + (1-\beta) * 0 = U_3(R,\beta) \\ 2-\beta &\geq 4\beta \\ 2 &\geq 5\beta \\ \frac{2}{5} &\geq \beta. \end{aligned}$$

(e) Find a Nash equilibrium that is not a Subgame Perfect or Weak Sequential Equilibrium of this game.

Since it must not be a SPE we should have 1 choose r. Now R is never a best response, so have 3 choose L. If 2 chooses O this is not a SPE because 1 is not taking the right action at an unreached decision. There is another one, however, if 2 chooses I then 1 will choose T. Thus there are two NE that are not SPE, specifying either one will be fine.

19. Consider the following signalling game. At the beginning of the game the baby bird (chick, C) is hungry (H) with probability $q > 0$ or full (F) with probability $1-q > 0$. Then the chick can squawk (S, make a lot of noise) or keep quiet (Q). Finally the parent can decide whether to give the chick the food (G) or not (N). If the chick squawks its payoffs are discounted

by β, $0 < \beta < 1$. The extensive form game can be represented as:



(a) Solve this game as if the parent knows whether the chick is hungry or not.

Player 2 will always give to hungry chicks and not to full ones. Player 1 will then always keep quiet.

(b) Find weak sequential equilibria such that the strategy of the chicks is as you found in 19a. Note that the parent's strategy will change with q.

Solution 5 So the strategy of the chicks is $(Q(H), Q(F))$ the best response of the parent depends on q:

$$\begin{aligned} U_2(G,Q) &= 3q + 2(1-q) = 2 + 2q \\ U_2(N,Q) &= 1q + 3(1-q) = 3 - 2q \end{aligned}$$

So $G(Q)$ is a best response if

$$\begin{aligned} 2 + 2q &\geq 3 - 2q \\ q &\geq \frac{1}{4} \end{aligned}$$

otherwise $N(Q)$ is the best response.

If the strategy is $G(Q)$ the response to S is unimportant, so this can be paired with either $G(S)$ or $N(S)$, thus the full strategy is either $(G(Q), G(S))$ or $(G(Q), N(S))$

If the strategy is $N(Q)$ then the response to S should be $N(S)$, or otherwise hungry chicks will squawk. Thus the strategy is $(N(Q), N(S))$

(c) Find the values of β such that there is a weak sequential equilibrium where parents only feed squawking chicks and only hungry chicks get fed.

The strategy specified is $(G(S), N(Q))$ *and we must find a condition such that the best response of the chicks is* $(S(H), Q(F))$*.* $S(H)$ *is always the best response to* $(G(S), N(Q))$ *since* $\beta > 0$*.* $Q(F)$ *is a best response if:*

$$\begin{aligned} 2\beta &\leq 1 \\ \beta &\leq \frac{1}{2} \end{aligned}$$

(d) Find the values of β and q such that there is a weak sequential equilibrium where parents only feed squawking chicks but all chicks get fed. Is this equilibrium Pareto Efficient?

Solution 6 The strategy must be $(G(S), N(Q))$, $(S(H), S(F))$, this requires that $2\beta \geq 1$ or $\beta \geq \frac{1}{2}$ and it must be a best response to feed all chicks:

$$\begin{aligned} U_2(G, S) &= 2q + (1-q) = 1 + q \\ U_2(N, S) &= q + 2(1-q) = 2 - q \end{aligned}$$

so $q \geq \frac{1}{2}$ is necessary.

This equilibrium is not Pareto efficient, it is Pareto dominated by the equilibrium $(Q(H), Q(F))$ $(G(Q), G(S))$, which has weaker existence conditions.

(e) Relate your finding about β above to when a signal can be effective.

If β *is too large then squawking is not a good signal because the full will do it just to get food. Thus* β *has too be low for it to work as a signal, but this is equivalent to saying that the signal must be costly in order to work.*

20. A firm and a union are negotiating over the wage of the workers. The firm's revenue is R and the wage bill it pays its workers is $W \geq 0$. The union is strong with probability $q > 0$ and weak with probability $1-q > 0$ and the firm does not know whether the union is strong or weak. If it is strong it will accept offers of W_h or higher ($R > W_h$), if it is weak it will accept offers of $W_l = 0$ or higher. In order to signal its strength it can go on strike for a length of time S. If it is strong the cost of going on strike for S length of time is $C_s(S) = c_s S$, if it is weak the cost is $C_w(S) = c_w S$ where $c_w > c_s > 0$. If the union rejects the offer both parties get zero. You may assume that the firm never offers any wage above W_h or below W_l.

(a) The structure of the game is that the union first goes on strike for a time S and then the firm makes a take it or leave it offer.

i. Find the set of equilibria where neither type of union goes on strike.

We must have that $S_w = S_s = 0$*, thus the firm is completely uninformed about the strength of the firm. Thus the firm will choose it's offer to maximize it's profits:*

$$\pi(W) = \begin{cases} R - W & \text{if} \quad W \geq W_h \\ (1-q)(R-W) & \text{if} \quad W_h > W \geq 0 \end{cases}$$

Thus it's offer will either be W_h *or* 0*, it will be* W_h *if*

$$\begin{aligned} R - W_h &\geq (1-q) R \\ W_h &\geq qR \end{aligned}$$

ii. Find the set of equilibria where both types of unions go on strike for the same length of time. (Hint: For some values of q the maximum length of the strike will be zero.)

We must have that $S_w = S_s = S^*$*, and the firm will still be still be completely uninformed about the strength of the firm. Thus the firm will choose* W_h *if* $W_h \leq qR$*,* 0 *else. If the firm chooses* 0 *then the optimal strike length is zero. If it chooses* W_h *it can use a strategy like:*

$$W(S) = \begin{cases} W_h & \text{if} \quad S \geq S^* \\ 0 & \text{if} \quad S < S^* \end{cases}$$

And the types of union will strike if

$$\begin{aligned} W_h - c_s S^* &\geq 0 \\ W_h - c_w S^* &\geq 0 \end{aligned}$$

and clearly the binding condition is $\frac{W_h}{c_w} \geq S^*$*.*

iii. Find the set of equilibria where the strong union goes on strike for longer than the weak union. How long will the weak union go on strike for?

The strategy of the firm is:

$$W(S) = \begin{cases} W_h & \text{if} \quad S \geq S^* \\ 0 & \text{if} \quad S < S^* \end{cases}$$

and the weak union will not go on strike since no matter what they do they will get 0*. The strong union will go on strike for* S^* *periods if*

$$W_h - c_s S^* \geq 0$$

the weak union will not go on strike if

$$W_h - c_w S^* \leq 0$$

requiring

$$\frac{W_h}{c_s} \geq S^* \geq \frac{W_h}{c_w}$$

(b) Now assume that the uncertainty is symmetric, or that both the union and the firm do not know if the union is weak or strong. However there are now two rounds of negotiations. First the union goes on strike for some length of time S_1 then the firm makes a take it or leave it offer. If the union does not accept the offer it can go on strike for some length of time S_2, and then the firm makes a final take it or leave it offer. You should assume that S_1 and S_2 can only be positive integer quantities ($\{0, 1, 2, ...\infty\}$). After the first strike the union knows whether it is weak or strong. *Clarification: If $S_1 = 0$ then the union does not know what type it is. If $S_1 = 0$ and a strong union is offered a wage lower than W_h the union will not accept it.*

  i. Find the set of equilibria where neither type of union goes on strike.
  It will be the same as above for the same case.

  ii. Now find the set of equilibria where the total time both unions go on strike ($S_1 + S_2$) is the same and $S_1 > 0$. (Notice that these equilibria will only exist for some values of q). When will the unions prefer these equilibria to the equilibria where neither type goes on strike?
  It will be the same as above, with S replaced with $S_1 + S_2$. The unions will never prefer these equilibria, because the outcomes will always be the same as if they had not gone on strike.

  iii. Now find the set of equilibria where the strong union goes on strike for longer than the weak union. What is the minimum and maximum value for S_1 in such an equilibrium?
  First of all, $S_1 > 0$ is necessary. A strategy for the firm is:

$$W(S) = \begin{cases} W_h & \text{if} \quad S_2 \geq S_2^* \text{ and } S_1 \geq S_1^* \\ 0 & \text{else} \end{cases}$$

  and obviously the weak firm will have $S_2 = 0$. Thus the equilibrium conditions for S_2 are that:

$$\begin{aligned} W_h - c_s (S_2^* + S_1^*) &\geq -c_s S_1^* \\ W_h - c_w (S_2^* + S_1^*) &\leq -c_w S_1^* \end{aligned}$$

  notice the S_1^* drops out of this equation, so the conditions are:

$$\frac{W_h}{c_s} \geq S_2^* \geq \frac{W_h}{c_w}$$

  The binding constraint on S_1^* is that a firm who does not know their type must be willing to strike for S_1^* periods to learn it.

$$\begin{aligned} q (W_h - c_s (S_2^* + S_1^*)) + (1-q)(-c_w S_1^*) &\geq 0 \\ q (W_h - c_s S_2^*) &\geq [q c_s + (1-q) c_w] S_1^* \\ \frac{q (W_h - c_s S_2^*)}{q c_s + (1-q) c_w} &\geq S_1^* \end{aligned}$$

notice that this also puts a tighter bound on $W_h - c_s S_2^*$ *than we have above, if* $\frac{W_h}{c_s} = S_2^*$ *then* $S_1^* = 0$*, which is not feasible. The actual upper bound on* S_2^* *is:*

$$\frac{W_h}{c_s} - \frac{1}{qc_2}\left(qc_s + (1-q)\,c_w\right) \geq S_2^*$$

5 Chapter 14 and 15

1. Consider the following Normal form game as the stage game of an infinitely repeated game with a discount factor δ, where $0 \leq \delta < 1$.

		Player 2		
		α	β	γ
	A	$-1; 8$	$\underline{5; 12}^{2*}$	$\underline{6; 6}^{*}$
Player 1	B	$0; -6$	$\underline{8; 0}_{2}^{12}$	$8; -3$
	C	$\underline{2; 2}_{1}^{12}$	$5; -1$	$\underline{9; 1}^{1*}$

(a) Find the best responses of both players in this stage game. You may mark them on the game but you will loose two points if you do not explain your notation below.

The best responses are denoted by a 1 (for player 1) or a 2 (for player 2) in the upper right hand corner.

(b) Find the pure strategy Nash equilibria of this stage game.

They are the boxes with both a 1 and 2 in the upper right hand corner.

(c) Define the *minimax* payoff in pure strategies for an arbitrary stage game G.

$$\underline{u}_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i\left(a_i, a_{-i}\right)$$

(d) Find the minimax payoff in pure strategies for both players in this stage game. Also write down the associated strategy pair.

The minimax payoff is denoted by a subscript of 1 *for player 1 and 2 for player 2. Note that it is always a Nash equilibrium.*

(e) Find the unique Pareto Efficient and symmetric strategy pair in this game, and then find all other Pareto Efficient strategy pairs in this game. *(Note: Your answer should be in terms of strategies, not payoffs.)*

These are denoted by a ∗ in the upper right hand corner. There are always at least three, and the points will be given for the argument for why these are Pareto Efficient. For each payoff you should argue that either another payoff is better for both or that there is no such payoff.

(f) Let $\vec{a}$ be the constant path where you play $a \in \{A, B, C\} \times \{\alpha, \beta, \gamma\}$ in every period. Prove that $V_i(\vec{a}) = \frac{1}{1-\delta} u_i(a)$ for $i \in \{1, 2\}$.

$$\begin{aligned}
V_i(\vec{a}) &= u_i(a) + \delta u_i(a) + \delta^2 u_i(a) + \delta^3 u_i(a) + \delta^4 u_i(a) + \delta^5 u_i(a) \ldots \\
(1-\delta) V_i(\vec{a}) &= u_i(a) + \delta u_i(a) + \delta^2 u_i(a) + \delta^3 u_i(a) + \delta^4 u_i(a) + \delta^5 u_i(a) \ldots \\
& \quad -0 - \delta u_i(a) - \delta^2 u_i(a) - \delta^3 u_i(a) - \delta^4 u_i(a) - \delta^5 u_i(a) \ldots \\
(1-\delta) V_i(\vec{a}) &= u_i(a) \\
V_i(\vec{a}) &= \frac{u_i(a)}{1-\delta}
\end{aligned}$$

this can also be done using the value function.

$$\begin{aligned}
V_i(\vec{a}) &= u_i(a) + \delta V_i(\vec{a}) \\
(1-\delta) V_i(\vec{a}) &= u_i(a) \\
V_i(\vec{a}) &= \frac{1}{1-\delta} u_i(a)
\end{aligned}$$

(g) Write down a strategy that will support the unique Pareto Efficient and symmetric strategy pair in this game, and find the minimal δ such that it is an equilibrium given your strategy. Be sure to prove that your strategy is an equilibrium in all subgames.

There are (by intention) two strategies that will do the job. The simple strategy I will denote the Grimm strategy: if $t = 1$ or a was played last period, play a this period, otherwise play the unique symmetric payoff Nash equilibrium (n). The more complex strategy is the mark II strategy: if $t = 1$ or a was played last period, play a this period, else if player 1 was the first to deviate play n, otherwise play the other Nash equilibrium. I will answer the question using the abstract payoffs.

for both strategies:

$$V_1(\vec{a}) = V_2(\vec{a}) = V(\vec{a}) = (a + \Delta) + \frac{\delta}{1-\delta}(a + \Delta)$$

For the grimm strategy,

$$V_1(\vec{n}) = V_2(\vec{n}) = V(\vec{n}) = \frac{1}{1-\delta} a$$

if player 1 deviates their best deviation is:

$$\hat{V}_1(\vec{a}) = a + \Delta + \max(c, d) + \frac{1}{1-\delta} a$$

if player 2 deviates their best deviation is:

$$\hat{V}_2(\vec{a}) = a + \Delta + b + \frac{1}{1-\delta} a$$

Thus 1 will follow the strategy if:

$$\begin{aligned}
a+\Delta+\max(c,d)-(a+\Delta) &\leq \frac{\delta}{1-\delta}\left((a+\Delta)-a\right)\\
\max(c,d) &\leq \frac{\delta}{1-\delta}\Delta\\
\max(c,d)(1-\delta) &\leq \delta\Delta\\
\max(c,d) &\leq \delta\left(\Delta+\max(c,d)\right)\\
\frac{\max(c,d)}{\Delta+\max(c,d)} &\leq \delta
\end{aligned}$$

:

$$\begin{aligned}
(12)-(6) &\leq \frac{\delta}{1-\delta}\left((6)-2\right)\\
6 &\leq \frac{\delta}{1-\delta}4\\
\frac{3}{5} &\leq \delta
\end{aligned}$$

Thus

$$\delta_G^* = \max\left(\frac{\max(c,d)}{4+\max(c,d)}, \frac{3}{5}\right)$$

with the Mark II strategy the incentives are identical for player 1, for player 2, let o be the other Nash equilibrium, then

$$V_2(\vec{o}) = \frac{1}{1-\delta}0$$

$$\hat{V}_2(\vec{a}) = 12 + \frac{1}{1-\delta}0$$

$$\begin{aligned}
(12)-(6) &\leq \frac{\delta}{1-\delta}(6)\\
6 &\leq \frac{\delta}{1-\delta}(6)\\
\frac{1}{2} &\leq \delta
\end{aligned}$$

thus

$$\delta_{II}^* = \max\left(\frac{\max(c,d)}{4+\max(c,d)}, \frac{1}{2}\right)$$

you will be graded based on whichever strategy you use.

You also need to cover the other subgames. In all these subgames independent of what a player does today they expect to play the same thing in the future. Thus the only thing that they need to care about are current incentives, and since they are always playing a Nash equilibrium these are always equilibria.

(h) Find the set of pure strategy pairs such that $a \in \{A, B, C\} \times \{\alpha, \beta, \gamma\}$ can be expected to be played in a subgame perfect equilibrium for high enough δ.

These are underlined in the games above. Again the number differs for different games, but the reasoning is the same for all games. Notice that three of these are known a-priori because they are Nash equilibria or the payoff in part g.

(i) Find a strategy that will support any of the strategy pairs you just said could be subgame perfect equilibria for high enough δ, and prove that for high enough δ any of these strategy pairs can be an equilibrium. Be sure to prove that your strategy is an equilibrium in all subgames. *Note: You do not have to solve the model in each and every case, just show that it can be an equilibrium.*

The strategy is the mark II strategy above, where a is now an arbitrary strategy in that group. To prove that they are an equilibrium, first of all if a NE is expected always in the future behavior today has no impact on the future, thus players should optimize their current payoff, and since they are playing a NE today that is being done. Second, the value of following the strategy is now:

$$V_i(\vec{a}) = u_i(a) + \frac{\delta}{1-\delta} u_i(a)$$

Define

$$\hat{u}_i(a) = \max_{b \in A_i} u_i(b, a_{-i})$$

then for player 2:

$$\hat{V}_2(\vec{a}) = \hat{u}_2(a) + \frac{\delta}{1-\delta} 0$$

and he will cooperate if:

$$\begin{aligned} \hat{u}_2(a) - u_2(a) &\leq \frac{\delta}{1-\delta}(u_2(a) - 0) \\ \frac{\hat{u}_2(a) - u_2(a)}{u_2(a)\,0} &\leq \frac{\delta}{1-\delta} \end{aligned}$$

since the right hand side converges to infinity and the left hand side is positive if $u_2(a) > 0$ (or equal and it is a Nash equilibrium) there is a δ_2^* that makes this an equilibrium.

For player 1

$$\hat{V}_1(\vec{a}) = \hat{u}_1(a) + \frac{\delta}{1-\delta} 2$$

$$\begin{aligned} \hat{u}_1(a) - u_1(a) &\leq \frac{\delta}{1-\delta}(u_1(a) - 2) \\ \frac{\hat{u}_1(a) - u_1(a)}{u_1(a) - 2} &\leq \frac{\delta}{1-\delta} \end{aligned}$$

Thus as long as $u_1(a) > 2$ (or equal and a is a Nash equilibrium) this is satisfied for large enough δ.

2. Consider the following stage game in an infinitely repeated game with a discount factor of $\delta \in [0, 1)$:

		Player 2 α	β	γ
	A	$2; 0$	$7; 6$	$2; 7^{12}$
Player 1	B	$7; 2^{12}$	$8; 1$	$-3; -3$
	C	$6; 6$	$11; 6^{1}$	$1; 11^{2}$

(a) Find the best responses for both players, you may mark them on the table above but you must explain your notation below.

I write a 1 in the upper right hand corner if it is a best response for 1, a 2 if it is a best response for 2.

(b) Find the pure strategy Nash equilibria.

These are the strategies that are best responses for both players, so these are (B, α) and (A, γ) in this game.

(c) Prove that the value of getting x every period in the future is $V(x) = x/(1-\delta)$.

(d) Find a subgame perfect equilibrium strategy and the minimal δ (call it δ^*) such that playing c_1 forever is the equilibrium path. *Be careful to prove it is an equilibrium in every subgame. Notice that 3 points will be given for correctly writing down the strategy.*

There are two strategies that can cause players to play c_1 forever. The first one is the simple trigger strategy: c_1 if $t = 1$ or in the last period players played c_1, otherwise (B, α). There is a more complicated one that gains us nothing because player 1 will never want to deviate. This is: c_1 if $t = 1$ or in the last period players played c_1, play (B, α) if player 2 was the first to deviate, otherwise play (A, γ).

Both of these will be a subgame perfect equilibrium when players are not playing c_1 because players always play a Nash equilibrium and the current behavior will never affect the future. Thus we only need to check the subgame where players expect to play c_1 forever. We will first check it for player 2 because player 1 is always best responding.

$$\begin{aligned} V_2^* &= \frac{1}{1-\delta} u_2(c_1) \\ &= \frac{\delta}{1-\delta} u_2(c_1) + \frac{1-\delta}{1-\delta} u_2(c_1) \\ &= \frac{\delta}{1-\delta} u_2(c_1) + u_2(c_1) \end{aligned}$$

$$\hat{V}_2 = u_2(C, \gamma) + \delta \frac{1}{1-\delta} u_2(B, \alpha)$$

$$
\begin{aligned}
V_2^* &\geq \hat{V}_2 \\
\frac{\delta}{1-\delta} u_2(c_1) + u_2(c_1) &\geq u_2(C,\gamma) + \frac{\delta}{1-\delta} u_2(B,\alpha) \\
\frac{\delta}{1-\delta}\left(u_2(c_1) - u_2(B,\alpha)\right) &\geq u_2(C,\gamma) - u_2(c_1) \\
\frac{\delta}{1-\delta}(6-2) &\geq (11) - (6) \\
\frac{\delta}{1-\delta} 4 &\geq 5 \\
\delta d &\geq 5 - \delta * 5 \\
\delta(9) &\geq (5) \\
\delta &\geq \frac{5}{9}
\end{aligned}
$$

and thus $\delta^* = \frac{5}{9}$. *Checking player 1:*

$$
\begin{aligned}
V_1^* &= \frac{1}{1-\delta} u_1(c_1) \\
\hat{V}_1 &= u_1(c_1) + \frac{\delta}{1-\delta} u_1(B,\alpha)
\end{aligned}
$$

and since $u_1(c_1) > u_1(B,\alpha)$ *this will be true for all* δ.

(e) Find a subgame perfect equilibrium strategy and the minimal δ (call it δ^*) such that playing c_2 forever is the equilibrium path. *Be careful to prove it is an equilibrium in every subgame. Notice that 3 points will be given for correctly writing down the strategy.*

In this case we must use the strategy: c_2 *if* $t = 1$ *or in the last period players played* c_2, *play* (B,α) *if player 2 was the first to deviate, otherwise play* (A,γ).

Like before if players do not expect to play c_2 *forever this is a subgame perfect equilibrium because actions today do not affect the future and players are playing a static Nash equilibrium.*

$$
\begin{aligned}
V_1^* &= u_1(c_2) + \frac{\delta}{1-\delta} u_1(c_2) \\
V_2^* &= u_2(c_2) + \frac{\delta}{1-\delta} u_2(c_2)
\end{aligned}
$$

$$
\begin{aligned}
\hat{V}_1 &= u_1(B,\alpha) + \frac{\delta}{1-\delta} u_1(A,\gamma) \\
\hat{V}_2 &= u_2(C,\gamma) + \frac{\delta}{1-\delta} u_2(B,\alpha)
\end{aligned}
$$

$$
\begin{aligned}
V_1^* &\geq \hat{V}_1 \\
u_1(c_2) + \frac{\delta}{1-\delta} u_1(c_2) &\geq u_1(B,\alpha) + \frac{\delta}{1-\delta} u_1(A,\gamma) \\
\frac{\delta}{1-\delta}\left(u_1(c_2) - u_1(A,\gamma)\right) &\geq u_1(B,\alpha) - u_1(c_2) \\
\frac{\delta}{1-\delta}\left((6)-(2)\right) &\geq (7)-(6) \\
\frac{\delta}{1-\delta} 4 &\geq 1 \\
\delta &\geq \frac{1}{5}
\end{aligned}
$$

$$
\begin{aligned}
V_2^* &\geq \hat{V}_2 \\
u_2(c_2) + \frac{\delta}{1-\delta} u_2(c_2) &\geq u_2(C,\gamma) + \frac{\delta}{1-\delta} u_2(B,\alpha) \\
\frac{\delta}{1-\delta}\left(u_2(c_2) - u_2(B,\alpha)\right) &\geq u_2(C,\gamma) - u_2(c_2) \\
\frac{\delta}{1-\delta}\left((5)-(2)\right) &\geq (11)-(5) \\
\frac{\delta}{1-\delta} 4 &\geq 5 \\
\delta &\geq \frac{4}{9}
\end{aligned}
$$

thus $\delta^* = \max\left(\frac{1}{5}, \frac{5}{9}\right)$*, and one can establish that* $\frac{5}{9} > \frac{1}{5}$ *thus the critical* δ^* *is* $\frac{5}{9}$*.*

(f) For every pair of pure strategies state whether there can or can not be a subgame perfect equilibrium for high enough δ where this pair of strategies is played forever on the equilibrium path. *You do not have to find the* δ*, simply state whether it is possible or not.*

First of all it is fairly easy to verify that the minimax payoffs for both players is b. Given this we can look at the game:

		Player 2		
		α	β	γ
	A	$2;0$	$7;6$	$2;7^{12}$
Player 1	B	$7;2^{12}$	$8;1$	$-3;-3$
	C	$6;6$	$11;6^{1}$	$1;11^{2}$

And see that $(A,\beta), (C,\alpha), (C,\beta)$ *can be supported because both players get strictly more than their minimax. Obviously* (B,α) *and* (A,γ) *can be because they are static Nash equilibria. Thus the strategy pairs* (A,α),(B,β),(B,γ), (C,γ) *can not be supported.*

3. Consider the repeated game with a Cournot Duopoly game as its stage game. Two firms each simultaneously choose to produce $q_i \geq 0$. The

market price is $P = 24-\frac{1}{2}(q_1+q_2)$, and each firm has the same symmetric costs, $c(q_i) = 12q_i$. In this question I do not promise that all answers will be in integers, they should be simple fractions however.

(a) Find the unique Nash equilibrium of this stage game, including the profits.

$$\max_{q_1} \left(24 - \frac{1}{2}(q_1+q_2)\right) q_1 - 12 * q_1$$

$$\left(24 - \frac{1}{2}(q_1+q_2)\right) - \frac{1}{2}q_1 - 12 = 0$$

Checking for a symmetric equilibrium:

$$\begin{aligned} \left(24 - \frac{1}{2}(q+q)\right) - \frac{1}{2}q - 12 &= 0 \\ q &= 8 \end{aligned}$$

since this is an equilibrium and I told you it is unique it must be the only equilibrium.

$$\begin{aligned} \pi_i &= \left(24 - \frac{1}{2}\left(\frac{2}{3}(24-12) + \frac{2}{3}(24-12)\right)\right)\frac{2}{3}(24-12) - 12\frac{2}{3}(24-12) \\ \pi_i &= \frac{2}{9}(12)^2 = 32 \end{aligned}$$

(b) Find the amount each would produce in the symmetric collusive outcome—where firms maximize the sum of the profits. Find the profits.

$$\max_{q_1,q_2} \left(24 - \frac{1}{2}(q_1+q_2)\right) q_1 - 12q_1 + \left(24 - \frac{1}{2}(q_1+q_2)\right) q_2 - 12q_2$$

$$\max_{q_1,q_2} \left(24 - \frac{1}{2}(q_1+q_2)\right)(q_1+q_2) - 12(q_1+q_2)$$

since only q_1+q_2 appear in this equation we can write $Q = q_1+q_2$ and proceed from there.

$$\max_{Q} \left(24 - \frac{1}{2}Q\right) Q - 12Q$$

$$\begin{aligned} \left(24 - \frac{1}{2}Q\right) - \frac{1}{2}Q - 12Q &= 0 \\ Q &= (24-12) \\ q &= \frac{Q}{2} = \frac{1}{2}(12) = 6 \end{aligned}$$

$$\pi_i = \left(24 - \frac{1}{2}\left(\frac{1}{2}(24-12) + \frac{1}{2}(24-12)\right)\right)\frac{1}{2}(24-12) - 12\frac{1}{2}(24-12)$$
$$\pi_i = \frac{1}{4}(24-12)^2 = 36$$

(c) If firm 2 produces the symmetric collusive output what is the optimal amount for firm one to produce? What is it's profits?

$$\max_{q_1}\left(24 - \frac{1}{2}\left(q_1 + \frac{1}{2}(24-12)\right)\right)q_1 - 12q_1$$
$$\max_{q_1}\frac{1}{4}q_1(3*24 - 3*12 - 2q_1)$$

$$(3*24 - 3*12 - 2q_1) - 2q_1 = 0$$
$$q_1 = \frac{3}{4}(24-12) = 9$$

$$\pi_1 = \left(24 - \frac{1}{2}\left(\frac{3}{4}(24-12) + \frac{1}{2}(24-12)\right)\right)\frac{3}{4}(24-12) - 12 * \frac{3}{2}(24-12)$$
$$= \frac{9}{32}(24-12)^2 = \frac{81}{2}$$

(d) Find the minmax profits and the minmax strategies in this game. Is it a Nash equilibrium? Prove your answer.

$$\max_{q_2}\min_{q_1}\left(24 - \frac{1}{2}(q_1+q_2)\right)q_2 - 12q_2 = \min_{q_1}\left(24 - \frac{1}{2}\left(q_1 + 12 - \frac{1}{2}q_1\right)\right)\left(12 - \frac{1}{2}q_1\right) - 12\left(\right.$$
$$= \min_{q_1}\frac{1}{2}\left(12 - 24 + \frac{1}{2}q_1\right)^2$$

$$12 - 24 + \frac{1}{2}q_1 = 0$$
$$q_1 = 24$$

another way to proceed is to notice that if $q_2 = 0$ *then profits must be equal to zero, which is the lower bound on firm 2's profits, and this can be achieved when:*

$$0 = 12 - \frac{1}{2}q_1$$
$$q_1 = 24$$

now I need to show this is not a Nash equilibrium. Obviously the firm being minmaxed is best responding, so let us check firm 1's profits:

$$\max_{q_1}\left(24 - \frac{1}{2}(q_1+0)\right)q_1 - 12q_1 = \max_{q_1}\left(24 - \frac{1}{2}q_1\right)q_1 - 12q_1$$

$$\begin{aligned} 24 - q_1 - 12 &= 0 \\ q_1 &= 12 \end{aligned}$$

which is clearly not the same as 6.

(e) Now consider the finitely repeated game, where this stage game is repeated T times. Find the unique subgame perfect equilibrium for all $0 \leq \delta \leq 1$. Prove your answer.

The unique subgame perfect equilibrium is $(8, 8)$ *repeated* T *times. To show this first consider the final period, then in this period the only equilibrium output is* $(8, 8)$ *(as I said there is a unique equilibrium of the stage game.) Now consider* $t < T$, *assuming this is true from* $t + 1$ *on. In this period the future will be the same no matter what is done today, and since there is a unique equilibrium of the stage game it must be what happens today.*

(f) Now consider the infinitely repeated game with this game as a stage game. Find a trigger or Grimm strategy that will support firms producing the symmetric collusive outcome in every period, and find the minimal δ such that firms can produce the symmetric collusive outcome every period. Be certain to carefully prove your strategy is an equilibrium. *The strategy is:*

$$q_t = \begin{cases} (6, 6) & \text{if} \quad q_{t-1} = (6, 6) \\ (8, 8) & \text{else} \end{cases}$$

This game has two subgames, one where players expect $(6, 6)$ *forever and one where players expect* $(8, 8)$ *forever.*

(2 points) in the subgame where they expect $(8, 8)$ *forever they will follow the strategy because no action today will affect the future strategy and the action today is a Nash equilibrium.*

(2 points) in the subgame where they expect $(6, 6)$ *we have to calculate the expected payoffs of following the strategy and an optimal deviation.*

$$\begin{aligned} v(s^*) &= \frac{1}{4}(24 - 12)^2 + \frac{\delta}{1 - \delta}\frac{1}{4}(24 - 12)^2 = \left(\frac{1}{8} + \frac{\delta}{1 - \delta}\frac{1}{8}\right) 2 * (12)^2 \\ \max_{s \neq s^*} v(s) &= \frac{9}{32}(24 - 12)^2 + \frac{\delta}{1 - \delta}\frac{2}{9}(a - c)^2 = \left(\frac{9}{64} + \frac{\delta}{1 - \delta}\frac{1}{9}\right) 2 * (12)^2 \end{aligned}$$

$$
\begin{aligned}
v(s^*) &\geq \max_{s \neq s^*} v(s) \\
\frac{1}{8} + \frac{\delta}{1-\delta}\frac{1}{8} &\geq \frac{9}{64} + \frac{\delta}{1-\delta}\frac{1}{9} \\
\frac{\delta}{1-\delta}\left(\frac{1}{8} - \frac{1}{9}\right) &\geq \frac{9}{64} - \frac{1}{8} \\
\frac{\delta}{1-\delta}\frac{1}{72} &\geq \frac{1}{64} \\
\frac{\delta}{1-\delta} &\geq \frac{9}{8} \\
\delta &\geq \frac{9}{17} = 0.529\,41
\end{aligned}
$$

notice this result does depends only on the linearity of demand and constant marginal cost.

(g) Is it possible that the collusive outcome could be supported for a lower δ in a subgame perfect equilibrium? If so how would you do this? Precise mathematical answers are not needed, just a discussion.
Yes, since the minmax in this game is not a Nash equilibrium it should be possible to use finite punishments and get cooperation where the punishment is the minmax payoffs. Whether these strategies give a lower δ or not is a subject for analysis. Just because I know how to let me specify the strategy and try to solve for the minimal such δ.

$$
a_t = \begin{cases} (8,8) & \text{if everyone has followed the strategy for the last } \hat{t} \text{ periods} \\ (24,0) & \text{if person 2 was the last person to deviate, and deviated in the last } \hat{t} \text{ periods} \\ (0,24) & \text{if person 1 was the last person to deviate, and deviated in the last } \hat{t} \text{ periods} \end{cases}
$$

$$
\begin{aligned}
v(s^*) &= \frac{1}{4}(24-12)^2 + \delta\frac{1}{4}(24-12)^2 + ... + \delta^{\hat{t}}\frac{1}{4}(24-12)^2 + \frac{\delta^{\hat{t}+1}}{1-\delta}\frac{1}{4}(24-12)^2 \\
\max_{s \neq s^*} v(s) &= \frac{9}{32}(24-12)^2 + \frac{\delta^{\hat{t}+1}}{1-\delta}\frac{1}{4}(24-12)^2
\end{aligned}
$$

$$
\begin{aligned}
v(s^*) &\geq \max_{s \neq s^*} v(s) \\
\frac{1}{4}(24-12)^2 + \delta\frac{1}{4}(24-12)^2 + ... + \delta^{\hat{t}-1}\frac{1}{4}(24-12)^2 + \frac{\delta^{\hat{t}}}{1-\delta}\frac{1}{4}(24-12)^2 &\geq \frac{9}{32}(24-12)^2 + \\
\frac{1}{4}(24-12)^2 + \delta\frac{1}{4}(24-12)^2 + ... + \delta^{\hat{t}-1}\frac{1}{4}(24-12)^2 &\geq \frac{9}{32}(24-12)^2 \\
\frac{1}{8} + \delta\frac{1}{8} + ... + \delta^{\hat{t}-1}\frac{1}{8} &\geq \frac{9}{64} \\
\frac{1-\delta^{\hat{t}}}{1-\delta}\frac{1}{8} &\geq \frac{9}{64} \\
\frac{1-\delta^{\hat{t}}}{1-\delta} &\geq \frac{9}{8}
\end{aligned}
$$

Now, however, we need to consider the subgames where person 2 has deviated and there are t periods of punishment left:

$$\begin{aligned} v(s^*,t) &= \delta^{t-1}\frac{1}{4}(24-12)^2+\delta^t\frac{1}{4}(24-12)^2+...+\frac{\delta^{\hat{t}}}{1-\delta}\frac{1}{4}(24-12)^2=\delta^{t-1}\frac{1-\delta^{\hat{t}-t-1}}{1-\delta}\frac{1}{4} \\ \max_{s\neq s^*} v(s,t) &= \frac{1}{2}(24-12)^2+\frac{\delta^{\hat{t}}}{1-\delta}\frac{1}{4}(24-12)^2 \end{aligned}$$

$$\begin{aligned} v(s^*,t) &\geq \max_{s\neq s^*} v(s,t) \\ \delta^{t-1}\frac{1-\delta^{\hat{t}-t-1}}{1-\delta}\frac{1}{4}(24-12)^2+\frac{\delta^{\hat{t}}}{1-\delta}\frac{1}{4}(24-12)^2 &\geq \frac{1}{2}(24-12)^2+\frac{\delta^{\hat{t}}}{1-\delta}\frac{1}{4}(24-12)^2 \\ \delta^{t-1}\frac{1-\delta^{\hat{t}-t-1}}{1-\delta}\frac{1}{8} &\geq \frac{1}{4} \\ \delta^{t-1}\frac{1-\delta^{\hat{t}-t-1}}{1-\delta} &\geq 2 \end{aligned}$$

and this is going to be hardest to solve when $t=\hat{t}-1$, or

$$\begin{aligned} \delta^{(\hat{t}-1)-1}\frac{1-\delta^{\hat{t}-(\hat{t}-1)-1}}{1-\delta} &\geq 2 \\ \frac{\delta^{\hat{t}-2}}{1-\delta} &\geq 2 \end{aligned}$$

so in order to solve this we need:

$$\frac{9}{8}\delta-\frac{1}{8}\geq\delta^{\hat{t}}\geq 2-2\delta$$

this interval will be non empty when:

$$\begin{aligned} \frac{9}{8}\delta-\frac{1}{8} &> 2-2\delta \\ \delta &> \frac{17}{25}=0.68 \end{aligned}$$

so it basically doesn't work in this game, but conceivably it could.

4. Consider the following game:

Player 2 (columns); Player 1 (rows)

	α	β	ψ	ε
A	14; 14	15; 13	5; 15[2]	4; 12
B	14; 13	17; 17[12]	−2; 10	0; 9
C	14; 8	9; 12	8; 17[12]	6; 14
D	16; 10[1]	10; 7	6; 9	11, 11[12]

(a) The Static Game:

  i. Find all the of the best responses for both players. *You will automatically loose 2 points if you do not explain your notation below.*

     I answered for each game above, and I put a 1 in the upper right hand corner if it was a best response for player 1, and a 2 if it was a best response for player 2.

  ii. Find all of the pure strategy Nash equilibria. For at least one of them explain why it is a Nash equilibrium.

     They have both a 1 and a 2 in the upper right hand corner, they are Nash equilibria because since both parties are best responding this means they are best responding given what the other person is doing. Thus they are rational and their beliefs are just that the person is taking the action they will take.

(b) The finite repeated game: This game is repeated twice, with the actions taken in the first period known before the second period actions are taken. The value of a sequence of action profiles is the sum of the stage game payoffs.

  i. In the second period which strategy profiles can be played in a subgame perfect equilibrium? Explain why these are the only strategy profiles that can be played.

     Any of the Nash equilibria of the static game, because since there is no future they must be maximizing today, and they must know what the other is doing. The only strategy profiles that withstand these criteria are Nash equilibria of the stateic game.

  ii. Write down a subgame perfect equilibrium strategy where (A, α) is played in the first period. Prove that your strategy is a subgame perfect equilibrium.

     There are actually two strategies, and I will specify both but only prove it for the easier one. To write these down I must first specify these strategies in a general form. Let h be the pareto efficient Nash equilibrium, m^1 be the one that gives player 1 his lowest Nash equilibrium payoff, and m^2 be the one that gives player 2 his worst Nash equilibrium payoff, notice this one also gives the same payoff to both players. The strategy I will prove the answer for is:

$$s_t = \begin{cases} (A, \alpha) & \text{if} \quad t = 1 \\ h & \text{if} \quad t = 2 \quad \text{and } (A, \alpha) \text{ was played last period} \\ m^2 & \text{if} \quad t = 2 \quad \text{else} \end{cases}$$

     the more (unecessarily) powerful strategy is:

$$s_t = \begin{cases} (A, \alpha) & \text{if} \quad t = 1 \\ h & \text{if} \quad t = 2 \quad \text{and } (A, \alpha) \text{ was played last period} \\ m^1 & \text{if} \quad t = 2 \quad \text{and } A \text{ was not played last period} \\ m^2 & \text{if} \quad t = 2 \quad \text{else} \end{cases}$$

the former is an equilibrium if

$$\begin{aligned}
\max_{a\in\{A,B,C,D\}} u_1(a,\alpha) - u_1(A,\alpha) &\leq u_1(h) - u_1\left(m^2\right) \\
\max_{a\in\{\alpha,\beta,\psi,\varepsilon\}} u_2(A,a) - u_2(A,\alpha) &\leq u_2(h) - u_2\left(m^2\right)
\end{aligned}$$

$$\begin{aligned}
16 - 14 &\leq 17 - 11 \text{ true} \\
15 - 14 &\leq 17 - 11 \text{ true}
\end{aligned}$$

iii. For all strategy pairs that are not either (A,α) or a Nash equilibrium of the static game show whether or not they can be equilibria of the twice repeated game.
For general $b = (b_1, b_2)$ to answer this correctly you need to consider the strategy:

$$s_t = \begin{cases} b & \text{if} \quad t = 1 \\ h & \text{if} \quad t = 2 \quad \text{and } b \text{ was played last period} \\ m^1 & \text{if} \quad t = 2 \quad \text{and } b_1 \text{ was not played last period} \\ m^2 & \text{if} \quad t = 2 \quad \text{else} \end{cases}$$

And then the inequalities are:

$$\begin{aligned}
\max_{a\in\{A,B,C,D\}} u_1(a,b_2) - u_1(b) &\leq u_1(h) - u_1\left(m^1\right) \\
\max_{a\in\{\alpha,\beta,\psi,\varepsilon\}} u_2(b_1,a) - u_2(b) &\leq u_2(h) - u_2\left(m^2\right)
\end{aligned}$$

	$u_1(h) - u_1\left(m^1\right)$	$u_2(h) - u_2\left(m^2\right)$				
	$17 - 8 = 9$	$17 - 11 = 6$				
	$u_1(a,b_2) - u_1(b)$	$u_2(b_1,a) - u_2(b)$	Eq?		$u_1(a,b_2) - u_1(b)$	u
(A,α)			ignore	(B,α)	$16 - 14 = 2$	1
(A,β)	$17 - 15 = 2$	$15 - 13 = 2$	yes	(B,β)		
(A,ψ)	$8 - 5 = 3$	$15 - 15 = 0$	yes	(B,ψ)	$8 - (-2) = 10$	1
(A,ε)	$11 - 4 = 7$	$15 - 12 = 3$	yes	(B,ε)	$11 - 0 = 11$	1
(C,α)	$16 - 14 = 2$	$17 - 8 = 9$	no	(D,α)	$16 - 16 = 0$	1
(C,β)	$17 - 9 = 8$	$17 - 12 = 5$	yes	(D,β)	$17 - 10 = 7$	1
(C,ψ)			ignore	(D,ψ)	$8 - 6 = 2$	1
(C,ε)	$11 - 6 = 5$	$17 - 14 = 3$	yes	(D,ε)		

i. The infinitely repeated game: The game is repeated for an infinite number of periods, with the actions taken in period $t-1$ known before actions in period t are taken. The value of a sequence of action profiles is the discounted sum of the stage game payoffs, with the discount factor $\delta \in (0,1)$. (Thus a payoff of x t periods in the future is worth $\delta^{t-1}x$ today).

ii. Show that for any x and $\delta \in (0,1)$ $\Sigma_{t=1}^{\infty}\delta^{t-1}x = \frac{1}{1-\delta}x$. *You may use this below even if you can not show this.*

$$\begin{aligned}
\Sigma_{t=1}^{\infty}\delta^{t-1}x &= x + \delta x + \delta^2 x + \delta^3 x + ... \\
(1-\delta)\,\Sigma_{t=1}^{\infty}\delta^{t-1}x &= (1-\delta)\left(x + \delta x + \delta^2 x + \delta^3 x + ...\right) \\
&= x + \delta x + \delta^2 x + \delta^3 x + ... \\
&\quad -\delta x - \delta^2 x - \delta^3 x - ... \\
(1-\delta)\,\Sigma_{t=1}^{\infty}\delta^{t-1}x &= x \\
\Sigma_{t=1}^{\infty}\delta^{t-1}x &= \frac{x}{1-\delta}
\end{aligned}$$

iii. Write down a subgame perfect equilibrium strategy where people expect to play (A,α) forever (assuming no one deviates). Find the minimal δ such that any strategy is an equilibrium. **Be sure to check all subgames two points will be given for checking the less obvious subgame.**
There are two such strategies:

$$s_t = \begin{cases} (A,\alpha) & \text{if} \quad t=1 \quad \text{or } (A,\alpha) \text{ was played last period} \\ m^2 & \text{else} \end{cases}$$

the more powerful strategy is:

$$s_t = \begin{cases} (A,\alpha) & \text{if} \quad t=1 \quad \text{or } (A,\alpha) \text{ was played last period} \\ m^1 & \text{if} \quad \text{P1 was the last person to deviate} \\ m^2 & \text{else} \end{cases}$$

for both of these I will first check the subgames where either m^1 or m^2 *is expected forever in the future. In these subgames the strategy is in equilibrium because* m^1 and m^2 *are Nash equilibria thus deviating will cost you today, plus it can only lower your payoff in the future. Thus it is an equilibrium.*
I will now check the subgame where players expect to play (A,α) forever in the future.

$$u_1(A,\alpha) + \frac{\delta}{1-\delta}u_1(A,\alpha) \geq \max_{a\in\{A,B,C,D\}} u_1(a,\alpha) + \frac{\delta}{1-\delta}u_1\left(m^i\right)$$

where $i \in \{1,2\}$.

$$\frac{\delta}{1-\delta}\left(u_1(A,\alpha) - u_1\left(m^i\right)\right) \geq \max_{a\in\{A,B,C,D\}} u_1(a,\alpha) - u_1(A,\alpha)$$

and for player 2 we likewise need to check:

$$\frac{\delta}{1-\delta}\left(u_2(A,\alpha) - u_2\left(m^2\right)\right) \geq \max_{a\in\{\alpha,\beta,\psi,\varepsilon\}} u_2(A,a) - u_1(A,\alpha)$$

$$\frac{\delta}{1-\delta}(14-11) \geq 16-14$$
$$\frac{\delta}{1-\delta}(14-8) \geq 16-14$$
$$\frac{\delta}{1-\delta}(14-11) \geq 15-14$$

The solution to the first equation is $\frac{2}{5}$, to the next two is $\frac{1}{4}$, so like before the solution is $\frac{1}{4}$.

iv. What is a minimax strategy? Find the minimax strategies and payoffs of both players.
For both players it is the strategy that gives them their worst Nash equilibrium payoff. You find this by minimizing a given player's payoffs assuming they best respond, and it so happens that in this game it is also the worst Nash equilibrium payoff.

v. For all strategy pairs that are not either (A,α) or a Nash equilibrium of the static game explain whether or not they can be equilibria of the infinitely repeated game as $\delta \to 1$. *Notice I am not requiring a rigorous proof, just an explanation will suffice.*
For any b what is required is that

$$\frac{\delta}{1-\delta}\left(u_1(b) - u_1\left(m^1\right)\right) \geq \max_{a\in\{A,B,C,D\}} u_1(a,b_2) - u_1(b)$$
$$\frac{\delta}{1-\delta}\left(u_2(b) - u_2\left(m^2\right)\right) \geq \max_{a\in\{\alpha,\beta,\psi,\varepsilon\}} u_2(b_1,a) - u_1(b)$$

and this will be satisfied as long as $u_1(b) - u_1\left(m^1\right) > 0$ *and* $u_2(b) - u_2\left(m^2\right) > 0$, *these strategey pairs are (for each game)*

		Player 2 α	β	ψ	ε
B Player 1	8	ignore	yes		
	B	yes	ignore		
	C		yes	ignore	
	D	yes			ignore

5. Consider the following Stage Game:

		Player 2 L	C	R
Player 1	U	4; 4	1; 2	6; 3
	M	7; 0	0; 3	3; 2
	D	3; 0	2; 1	2; 0

(a) Find all the best responses of both players and the Nash equilibrium (or equilibria) strategies. *You may use the table above to mark your answers but explain your notation below.*

		Player 2		
		L	C	R
	U	$4;4^2$	$1;2$	$6;3^1$
Player 1	M	$7;0^1$	$0;3^2$	$3;2$
	D	$3;0$	$2;1^{12}$	$2;0$

Like always I put a 1 in the upper right hand corner for player 1's best responses, and likewise for 2. The NE is the only square with a 1 and 2 in it.

(b) Consider this now as the stage game of a standard T period repeated game (where $T < \infty$.) Find all the equilibria of this game as $T \to \infty$, prove your answer.

For any finite T the only equilibrium is (D, C) regardless of history.

I will prove this is true by induction. First consider the final period, $t = T$. In this period their is no future so the only thing to do is optimize your current payoffs. This must result in a NE of the stage game being played and thus will result in (D, C) being played.

For $t = T - 1$ now in the next period (D, C) will be played no matter what. Thus the future is fixed, the only thing that can be changed is my current payoffs. Like in period T this will result in the Nash equilibrium, (D, C) being played.

For $t < T - 1$ assume by induction that it is true from $t + 1$ on. Then obviously, like before, the only thing to optimize is this period's payoffs, which will result in (D, C) being played. Thus the proof is done.

(c) Consider this now as the stage game of an infinitely repeated game, let the discount factor of both players be $\delta \in (0, 1)$.

i. Find a strategy such that players play (U, L) in every period. This strategy must be a subgame perfect equilibrium for high enough δ.

The strategy is: In period 1 play (U, L) in period $t > 1$ play (U, L) if (U, L) was played yesterday, otherwise play (D, C)

ii. Prove that this is an equilibrium strategy for high enough δ, and find the critical δ^* such that if $\delta \geq \delta^*$ then this is a subgame perfect equilibrium.

$$
\begin{aligned}
V_1^{eq} &= u_1(U, L) + \frac{\delta}{1-\delta} u_1(U, L) \\
V_1^{dev} &= u_1(M, L) + \frac{\delta}{1-\delta} u_1(D, C)
\end{aligned}
$$

$$\begin{aligned} V_1^{eq} &\geq V_1^{dev} \\ u_1(U,L) + \frac{\delta}{1-\delta} u_1(U,L) &\geq u_1(M,L) + \frac{\delta}{1-\delta} u_1(D,C) \\ \frac{\delta}{1-\delta}\left(u_1(U,L) - u_1(D,C)\right) &\geq u_1(M,L) - u_1(U,L) \\ \frac{\delta}{1-\delta} &\geq \frac{u_1(M,L) - u_1(U,L)}{u_1(U,L) - u_1(D,C)} = \frac{7-4}{4-2} = \frac{3}{2} \\ \delta &\geq \frac{3}{5} \end{aligned}$$

iii. Consider the following strategy: "If U last period then (U,L) this period, otherwise (D,C)." Explain in intuitive terms why this can not be an equilibrium for high enough δ, and show precisely that it can never be an equilibrium.
This strategy makes it possible for player 1 to rebuild his reputation by deviating a second time after a first deviation. After a deviation:

$$\begin{aligned} V_1^{eq} &= \frac{1}{1-\delta} 2 \\ V_1^{dev} &= u_1(U,C) + \frac{\delta}{1-\delta} u_1(U,L) \\ &= 1 + \frac{\delta}{1-\delta} 4 \end{aligned}$$

$$\begin{aligned} V_1^{eq} &\geq V_1^{dev} \\ \frac{1}{1-\delta} 2 &\geq 1 + \frac{\delta}{1-\delta} 4 \\ 2 &\geq (1-\delta) 1 + \delta 4 \\ 2 &\geq 3\delta + 1 \\ \delta &\leq \frac{1}{3} \end{aligned}$$

Since above we proved it was only an equilibrium to cooperate if $\delta \geq \frac{3}{5}$ *this is never an equilibrium*

iv. Which strategy pairs can be played every period in a subgame perfect equilibrium? Explain why.
The folk theorem tells us that any strategy that gives a strictly higher payoff than the minimax (the NE in this game) can be supported. Thus anything which gives strictly more than $u(D,C) = (2,1)$ *these are:* (U,L), (U,R), (M,R). *Furthermore obviously* (D,C) *can be supported.*

6. You are explaining the restaurant quality problem to another student (restaurants have a short run incentive to produce low quality). This

student says: "I can trust restaurants to produce high quality. If I don't like the food I am served I just won't go back for a while."

Explain in what situations this student would be correct, and when this student would not be correct. Further discuss what considerations should go into how long "a while" should be. ("a while" is the same as "for some period of time.")

There are three factors to consider:

(a) *How frequently they go to that restaurant (how patient the restaurant is or how near δ is to one.) Notice that when they say "that restaurant" they can really interpret it as "a restaurant of that chain" which is one of the reasons that chain restaurants are so successful.*

(b) *The amount it costs them to produce high quality. (Their benefit from producing low quality.)*

(c) *How much it costs them to lose this patron's business (A per period cost, basically the average profit of the restaurant from each visit.)*

Most of the points for the answer will be based on properly identifying these three important factors.

If this student does not go to the restaurant frequently enough then (a) fails and the student should not trust that restaurant. An example of this is eating at an unknown restaurant when travelling. Unless they make that trip routinely, they probably can not trust the restaurants quality. Of course if it is a chain restaurant (like Quick China, or Tadem Pizza) then they may be able to trust it because they frequent a restaurant ***in that chain*** *frequently.*

In general, the less frequently a restaurant is visited the longer they need to stay away—in terms of number of visits, not days. At some point any punishment they can impose is going to be insufficient.

If this student goes to this restaurant frequently enough. then when deciding on how long of "a while" is enough they need to think about the ratio of b to c. If b increases then they need to stay away longer, if c increases then they can go back more quickly.

Of course another important consideration is that if everyone uses the same strategy (and to a great extent we all do) then it may be that the punishments others can impose will be enough. To save money by producing low quality restaurants generally need to shut off (or raise the temperature in) the refrigerators or do something else that will affect ***every*** *customer. This explains, to a great extent, why tourist restaurants are often of lower quality than local restaurants. Why restaurants that rely on long distance travelers are often of poor quality. Simply speaking they do not have the same marginal incentive to produce high quality, and they don't.*

You also see them in very scary looking "fast food" restaurants on the highways. You may eat there happily, but... I usually look for a Kebap place.

I have extensive experience working in restaurants and you can take my word for it. A steam table is an ideal environment for germs to grow in.) Since historically Turks did not go to restaurants much the obvious benefit of food that was simple—and therefore easy to make trustworthily—is obvious. Of course this is just my crackpot theory, but do you have an alternative explanation for the obvious divide between what Turks eat at home and what they eat in traditional Turkish restaurants? I sincerely wish more "household cuisine" restaurants would open up. *This is also my theory of why Turkish restaurant cuisine is so much less varied than Turkish household cuisine. Restaurant cuisine is dominated by "pieces of meat cooked a long time over an open fire." The risk of food poisoning from that type of food is much lower than the risk in most household cuisine dishes—which are generally made a long time ahead of time and then reheated as needed. (Steam tables are very risky.[1]) Since historically Turks did not go to restaurants much the obvious benefit of food that was simple—and therefore easy to make trustworthily—is obvious. Of course this is just my crackpot theory, but do you have an alternative explanation for the obvious divide between what Turks eat at home and what they eat in traditional Turkish restaurants? I sincerely wish more "household cuisine" restaurants would open up.*

7. *Assume that the following stage game is repeated T times.*

	α	β	ψ
A	$0; 4$	$0; -2$	$5; 6$
B	$0; 5$	$6; 7$	$3; 2$
C	$2; 3$	$8; 1$	$0; -2$

..

   (a) *Find all the best responses of both players and the Nash equilibria of this stage game. Write the Nash equilibrium strategies below.*

   The NE are (C, α) and (A, ψ)

	α	β	ψ
A	$0; 4$	$0; -2$	$\underline{5; 6}^{12}$
B	$0; 5$	$6; 7^{2}$	$3; 2$
C	$\underline{2; 3}^{12}$	$8; 1^{1}$	$0; -2$

..

[1] What is a steam table? It is a large bowl or tray of food sitting in a bath of water, and the water is heated from below with some type of flame. They are very common on campus: Cicek, Speed, the Tablodot, and many other restaurants use them. Of course since they depend on repeat daily business I trust them implicitly.

You also see them in very scary looking "fast food" restaurants on the highways. You may eat there happily, but... I usually look for a Kebap place.

I have extensive experience working in restaurants and you can take my word for it. A steam table is an ideal environment for germs to grow in.

(b) *If $T = 2$.*

i. *Show that there is a Subgame Perfect equilibrium strategy where (B, β) can be the action pair that is played in the first period. Be careful to prove precisely that it is Subgame Perfect.*
Consider the strategy

$$\begin{aligned} a_1 &= (B, \beta) \\ a_2 &= \begin{cases} (A, \psi) & \text{if} \quad a_1 = (B, \beta) \\ (C, \alpha) & \text{else} \end{cases} \end{aligned}$$

This is a subgame perfect equilibrium in period 2 because the players always play a NE of the static game and there is no future.
This is a SPE in period 1 if

$$\begin{aligned} U_1(B, \beta) + U_1(A, \psi) &\geq \max_{a \in \{A,B,C\}} U_1(a, \beta) + U_1(C, \alpha) \\ 6 + 5 &\geq 8 + 2 \\ U_2(B, \beta) + U_2(A, \psi) &\geq \max_{a \in \{\alpha,\beta,\psi\}} U_2(B, a) + U_2(C, \alpha) \\ 7 + 6 &\geq 7 + 3 \end{aligned}$$

though checking the second inequality is unnecessary since β is the best response to B.

ii. *Find all action pairs that can be played in a Subgame Perfect equilibrium in the first period. Explain your answer.* (Note, one explanation should be sufficient for all the action pairs.)
Consider the strategy

$$\begin{aligned} a_1 &= (X, \xi) \\ a_2 &= \begin{cases} (A, \psi) & \text{if} \quad a_1 = (X, \xi) \\ (C, \alpha) & \text{else} \end{cases} \end{aligned}$$

then this an equilibrium if:

$$\begin{aligned} U_1(X, \xi) + U_1(A, \psi) &\geq \max_{a \in \{A,B,C\}} U_1(a, \xi) + U_1(C, \alpha) \\ U_1(A, \psi) - U_1(C, \alpha) &\geq \max_{a \in \{A,B,C\}} U_1(a, \xi) - U_1(X, \xi) \\ 3 &\geq \max_{a \in \{A,B,C\}} U_1(a, \xi) - U_1(X, \xi) \\ U_2(X, \xi) + U_2(A, \psi) &\geq \max_{a \in \{\alpha,\beta,\psi\}} U_2(X, a) + U_2(C, \alpha) \\ U_2(A, \psi) - U_2(C, \alpha) &\geq \max_{a \in \{\alpha,\beta,\psi\}} U_2(X, a) - U_2(X, \xi) \\ 3 &\geq \max_{a \in \{\alpha,\beta,\psi\}} U_2(X, a) - U_2(X, \xi) \end{aligned}$$

so now we just have to go through the possibilities for (X, ξ).

(X,ξ)	$\max_{a\in\{A,B,C\}} U_1(a,\xi) - U_1(X,\xi)$	$\max_{a\in\{\alpha,\beta,\psi\}} U_2(X,a) - U_2(X,\xi)$	*SPE?*
(A,α)	$2-0=2$	$6-4=2$	*Yes*
(A,β)	$8-0=8$	$6-(-2)=8$	***No***
(A,ψ)	$5-5=0$	$6-6=0$	*Yes*
(B,α)	$2-0=2$	$7-5=2$	*Yes*
(B,β)	$8-6=2$	$7-7=0$	*Yes*
(B,ψ)	$5-3=2$	$7-2=5$	***No***
(C,α)	$2-2=0$	$3-3=0$	*Yes*
(C,β)	$8-8=0$	$3-1=2$	*Yes*
(C,ψ)	$5-0=5$	$3-(-2)=5$	***No***

(c) *Find the minimal T such that (A,β) can be played in the first period of a Subgame Perfect Equilibrium.*

Consider the strategy

$$\begin{aligned} a_1 &= (A,\beta) \\ \textit{for } t &> 1 \\ a_t &= \begin{cases} (A,\psi) & \textit{if} \quad a_1 = (A,\beta) \\ (C,\alpha) & \textit{else} \end{cases} \end{aligned}$$

this is a SPE from period 2 on because no matter what is done it will not affect the future and the strategy calls for a NE of the static game. In period 1 this is an equilibrium if:

$$\begin{aligned} U_1(A,\beta) + (T-1)U_1(A,\psi) &\geq \max_{a\in\{A,B,C\}} U_1(a,\beta) + (T-1)U_1(C,\alpha) \\ (T-1)\left(U_1(A,\psi) - U_1(C,\alpha)\right) &\geq \max_{a\in\{A,B,C\}} U_1(a,\beta) - U_1(A,\beta) \\ (T-1)(5-2) &\geq 8-0 \\ T &\geq \frac{8}{3}+1 \\ U_2(A,\beta) + (T-1)U_2(A,\psi) &\geq \max_{a\in\{\alpha,\beta,\psi\}} U_2(A,a) + (T-1)U_2(C,\alpha) \\ (T-1)(6-3) &\geq 6-(-2) \\ T &\geq \frac{8}{3}+1 \end{aligned}$$

so if $T \geq 4$ this is an equilibrium.

8. *Consider the infinitely repeated game with the following stage game.*

	α	β	ψ
A	$2;3$	$-4;24$	$6;6$
B	$3;4$	$-3;8$	$4;1$
C	$6;-2$	$0;0$	$8;-1$

..

For a sequence of action pairs $\mathcal{A} = \{a_t\}_{t=1}^{\infty}$ the value of player i of this sequence is $v_i(\mathcal{A}) = \sum_{t=1}^{\infty} \delta^{t-1} u_i(a_t)$ where $\delta \in (0,1)$ and $u_i(a_t)$ is the utility of person i from the stage game when the action pair a_t is played.

(a) *Find all the best responses of both players and the Nash equilibria of this stage game. Write the Nash equilibria strategies below.*

	α	β	ψ
A	$2;3$	$-4;24^2$	$6;6$
B	$3;4$	$-3;8^2$	$4;1$
C	$6;-2^1$	$0;0^{12}$	$8;-1^1$

NE$=(C,\beta)$

(b) *Find a Subgame Perfect equilibrium strategy and the minimal δ such that (B,α) is the action pair that is played unless someone deviates from the strategy.*

Consider the strategy:

$$a_t = \begin{cases} (B,\alpha) & \textit{if always } (B,\alpha) \textit{ in the past} \\ (C,\beta) & \textit{else} \end{cases}$$

if players are supposed to play (C,β) this is a SPE because what they do will not affect the future and this is a NE of the stage game. If players are supposed to play (B,α) in the future then this is an equilibrium if:

$$\begin{aligned} \frac{1}{1-\delta} u_1(B,\alpha) &\geq \max_{a\in\{A,B,C\}} u_1(a,\alpha) + \frac{\delta}{1-\delta} u_1(C,\beta) \\ \frac{1}{1-\delta} 3 &\geq 6 + \frac{\delta}{1-\delta}(0) \\ 3 &\geq 6(1-\delta) \\ 6\delta &\geq 3 \\ \delta &\geq \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \frac{1}{1-\delta} u_2(B,\alpha) &\geq \max_{a\in\{\alpha,\beta,\psi\}} u_2(B,a) + \frac{\delta}{1-\delta} u_2(C,\beta) \\ \frac{1}{1-\delta} 4 &\geq 8 + \frac{\delta}{1-\delta}(0) \\ \delta &\geq \frac{1}{2} \end{aligned}$$

so this is a SPE if $\delta \geq \frac{1}{2}$.

(c) *Find a Subgame Perfect equilibrium strategy and the minimal δ such that (A,ψ) is the action pair that is played unless someone deviates from the strategy.*

Consider the strategy:

$$a_t = \begin{cases} (A,\psi) & \textit{if always } (A,\psi) \textit{ in the past} \\ (C,\beta) & \textit{else} \end{cases}$$

then this is an equilibrium if:

$$\begin{aligned} \frac{1}{1-\delta}u_1(A,\psi) &\geq \max_{a\in\{A,B,C\}} u_1(a,\psi) + \frac{\delta}{1-\delta}u_1(C,\beta) \\ \frac{1}{1-\delta}6 &\geq 8 + \frac{\delta}{1-\delta}(0) \\ 8\delta &\geq 2 \\ \delta &\geq \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \frac{1}{1-\delta}u_2(A,\psi) &\geq \max_{a\in\{\alpha,\beta,\psi\}} u_2(A,a) + \frac{\delta}{1-\delta}u_2(C,\beta) \\ \frac{1}{1-\delta}6 &\geq 24 + \frac{\delta}{1-\delta}(0) \\ 24\delta &\geq 18 \\ \delta &\geq \frac{3}{4} \end{aligned}$$

so $\delta \geq \frac{3}{4}$ is necessary for this to be an equilibrium.

(d) *Find all action pairs that can be the action pair played unless someone deviates from the strategy in a Subgame Perfect equilibrium.* (Note: one explanation should work for all action pairs)

The minimax payoff in this game is $(0,0)$ which is also the Nash equilibrium (in pure strategies) thus any pair of actions that gives strictly more than $(0,0)$ *is fine.* These are $(A,\alpha),(B,\alpha),(A,\psi),(B,\psi)$, *and of course* (C,β) will work because it is a NE of the stage game. .

9. *Consider the following stage game:*

	α	β	ψ	δ	ε
A	$9;3$	$4;4$	$4;2$	$\underline{3;5}^{12}$	$4;4$
B	$11;3^{1}$	$\underline{7;6}^{12}$	$2;5$	$2;4$	$0;0$
C	$0;1$	$5;1$	$4;-1$	$1;0$	$\underline{5;2}^{12}$
D	$10;9$	$6;10^{2}$	$3;-3$	$2;7$	$2;8$
E	$8;1$	$5;2$	$\underline{5;4}^{12}$	$2;1$	$4;3$

(a) *Find all of the best responses and Nash equilibria of this game.*

The best responses have a 1 or 2 in the upper right hand corner of the table above. The NE are all underlined, there are 4.

(b) Define the minimax *payoff, what is it's relationship with individual rationality? Find the minimax payoff for both players in this stage game.*

The minimax payoff is the amount a player can receive when all other players work to get him as low of a payoff as possible, or given that the other players are minimizing a given players payoff what is the maximum he can get. Mathematically it is:

$$m_i = \max_{a_i \in A_i} \min_{a_{-i} \in A_{-i}} u_i \left(a_i, a_{-i}\right)$$

For the row player the relevant payoffs are: $11\,(\alpha)$, $7\,(\beta)$, $5\,(\psi)$, $3\,(\delta)$, $5\,(\varepsilon)$, so the minimax strategy is (A, δ) and the minimax payoff is 3. For the column player the relevant payoffs are: $5\,(A)$, $6\,(B)$, $2\,(C)$, $10\,(D)$, $4\,(E)$, so the minimax strategy is (C, ε) and the minimax payoff is 2.

(c) Now consider the finitely repeated game where this stage game is repeated T times. In all cases you want to show that a pure strategy pair of actions (like (D, α)) can or can not be supported as the pair of actions in the first period of a Subgame Perfect equilibrium. In every case be sure to check all of the Subgames.

i. If $T = 2$ show that (D, α) can be the first period action profile.

There are several strategies that can do this. In all of them if people do what they are supposed to in the first period then in the second period they play (B, β), otherwise there is some punishment strategy. I will prove the results for only two of the 9 possible strategies. Either:

$$s_{1t} = \begin{cases} D & \textit{if} & & & t = 1 \\ B & & a_1 = (D, \alpha) & \textit{and} & t > 1 \\ E & \textit{else} & & & \end{cases} \quad s_{2t} = \begin{cases} \alpha & \textit{if} & & & t = 1 \\ \beta & & a_1 = (D, \alpha) & \textit{and} & t > 1 \\ \psi & \textit{else} & & & \end{cases}$$

or:

$$s_{1t} = \begin{cases} D & \textit{if} & & & t = 1 \\ B & \textit{if} & a_{t-1} = (D, \alpha) & \textit{and} & t > 1 \\ A & \textit{if} & a_{1,t-1} \neq D & \textit{and} & t > 1 \\ C & \textit{else} & & & \end{cases} \quad s_{2t} = \begin{cases} \alpha & \textit{if} & & & t = \\ \beta & \textit{if} & a_{t-1} = (D, \alpha) & \textit{and} & t > \\ \delta & \textit{if} & a_{1,t-1} \neq D & \textit{and} & t > \\ \varepsilon & \textit{else} & & & \end{cases}$$

Period 1, Player 1, first strategy:

$$10 + 7 = 17 > 16 = 11 + 5$$

Period 1, Player 2, first strategy:

$$9 + 6 = 15 > 14 = 10 + 4$$

Period 1, Player 1, second strategy:

$$10 + 7 = 17 > 14 = 11 + 3$$

Period 1, Player 2, second strategy:

$$9 + 6 = 15 > 12 = 10 + 2$$

In period 2 since there is no future and the actions called for are always NE of the stage game, players will follow the strategy.

ii. *If $T = 2$ show that (E, α) can be the first period action profile.*
The only strategy that will work is:

$$s_{1t} = \begin{cases} E & if & & & t = 1 \\ B & if & a_{t-1} = (D, \alpha) & and & t > 1 \\ A & if & a_{1,t-1} \neq D & and & t > 1 \\ C & else & & & \end{cases} \quad s_{2t} = \begin{cases} \alpha & if & & & t = \\ \beta & if & a_{t-1} = (D, \alpha) & and & t > \\ \delta & if & a_{1,t-1} \neq D & and & t > \\ \varepsilon & else & & & \end{cases}$$

Period 1, Player 1, first strategy:

$$8 + 7 = 15 > 14 = 11 + 3$$

Period 1, Player 2, first strategy:

$$2 + 6 = 8 > 7 = 5 + 2$$

In period 2 since there is no future and the actions called for are always NE of the stage game, players will follow the strategy.

iii. *If $T = 2$ find the three pairs of actions that* cannot *be first period action profile. Explain why they can not be.*
For both players the difference between the best NE and the worst NE is 4 points, thus the maximum difference between an action pair and the best response to that action pair must be 4. For example for player 1

$$\begin{aligned} u_1(c_1, c_2) + 7 &\geq \max_{a_1 \in A_1} u_1(a_1, c_2) + 3 \\ 4 &\geq \max_{a_1 \in A_1} u_1(a_1, c_2) - u_1(c_1, c_2) \end{aligned}$$

For player 2

$$\begin{aligned} u_2(c_1, c_2) + 6 &\geq \max_{a_2 \in A_2} u_2(c_1, a_2) + 2 \\ 4 &\geq \max_{a_2 \in A_2} u_2(c_1, a_2) - u_2(c_1, c_2) \end{aligned}$$

This is only rules out for the action pairs (C, α), (D, ψ), (B, ε).

iv. *What is the minimal T such that* every *pair of actions can be the first period action profile?*
To answer this we must use the general strategy:

$$s_{1t} = \begin{cases} c_1 & if & & & t = 1 \\ B & if & a_{t-1} = (D, \alpha) & and & t > 1 \\ A & if & a_{1,t-1} \neq D & and & t > 1 \\ C & else & & & \end{cases} \quad s_{2t} = \begin{cases} c_2 & if & & & t = \\ \beta & if & a_{t-1} = (D, \alpha) & and & t > \\ \delta & if & a_{1,t-1} \neq D & and & t > \\ \varepsilon & else & & & \end{cases}$$

$$\begin{aligned}
v_1\left((B,\beta)_+\right) &= (T-1)\,7 \\
v_1\left((A,\delta)_+\right) &= (T-1)\,3 \\
v_2\left((B,\beta)_+\right) &= (T-1)\,6 \\
v_2\left((C,\varepsilon)_+\right) &= (T-1)\,2
\end{aligned}$$

So for player 1 the inequality is:

$$\begin{aligned}
u_1(c_1,c_2)+(T-1)\,7 &\geq \max_{a_1\in A_1} u_1(a_1,c_2)+(T-1)\,3 \\
(T-1)\,4 &\geq \max_{a_1\in A_1} u_1(a_1,c_2)-u_1(c_1,c_2) \\
T &\geq \frac{\max_{a_1\in A_1} u_1(a_1,c_2)-u_1(c_1,c_2)}{4}+1
\end{aligned}$$

$$\max_{a_1\in A_1} u_1(a_1,c_2)-u_1(c_1,c_2)\in\{0,1,2,3,11,5\}$$

$$T\geq\frac{11}{4}+1=3.75$$

$$\begin{aligned}
u_2(c_1,c_2)+(T-1)\,6 &\geq \max_{a_2\in A_2} u_2(c_1,a_2)+(T-1)\,2 \\
(T-1)\,4 &\geq \max_{a_2\in A_2} u_2(c_1,a_2)-u_2(c_1,c_2) \\
T &\geq \frac{\max_{a_2\in A_2} u_2(c_1,a_2)-u_2(c_1,c_2)}{4}+1
\end{aligned}$$

$$\max_{a_2\in A_2} u_2(c_1,a_2)-u_2(c_1,c_2)\in\{0,1,2,3,13,6\}$$

$$T\geq\frac{13}{4}+1=4.25$$

so T must be 5 or higher.

(d) *Now consider the infinitely repeated game with the game above as the stage game. Let the common discount factor be δ. In all cases you want to find when a particular pure strategy action pair (like (D,α)) can be the action pair that people expect to always play as part of a Subgame Perfect equilibrium. In other words the actions they expect to play in the first period and every period thereafter. In every case be sure to check all of the Subgames.*

i. *Find a strategy and the minimal δ (given your strategy) such that (D,α) can be supported as equilibrium.*

Like before there are many such strategies, I will provide answers for: Either:

$$s_{1t}=\begin{cases} D & \text{if} \quad a_{t-1}=(D,\alpha) \\ E & \text{else}\end{cases} \qquad s_{2t}=\begin{cases} \alpha & \text{if} \quad a_{t-1}=(D,\alpha) \\ \psi & \text{else}\end{cases}$$

or:

$$s_{1t} = \begin{cases} D & \textit{if} \quad a_{t-1} = (D,\alpha) \\ A & \textit{if} \quad \textit{1 was the first to deviate} \\ C & \textit{else} \end{cases} \quad s_{2t} = \begin{cases} \alpha & \textit{if} \quad a_{t-1} = (D,\alpha) \\ \delta & \textit{if} \quad \textit{1 was the first to deviate} \\ \varepsilon & \textit{else} \end{cases}$$

With the first strategy, first player, if in the future one expects $(D,\alpha)_+$:

$$\begin{aligned} 10 + \frac{\delta}{1-\delta}10 &\geq 11 + \frac{\delta}{1-\delta}5 \\ \delta &\geq \frac{1}{6} \end{aligned}$$

second player, same subgame:

$$\begin{aligned} 9 + \frac{\delta}{1-\delta}9 &\geq 10 + \frac{\delta}{1-\delta}4 \\ \delta &\geq \frac{1}{6} \end{aligned}$$

if they expect (E,ψ) in the future then no action taken today will affect the future. That and the fact that (E,ψ) is a NE guarantee they are best responding.

ii. Find a strategy and the minimal δ (given your strategy) such that (A,ε) can be supported as equilibrium.
Only one strategy will work for this action pair:

$$s_{1t} = \begin{cases} A & \textit{if} \quad a_{t-1} = (A,\varepsilon) \\ A & \textit{if} \quad \textit{1 was the first to deviate} \\ C & \textit{else} \end{cases} \quad s_{2t} = \begin{cases} \varepsilon & \textit{if} \quad a_{t-1} = (A,\varepsilon) \\ \delta & \textit{if} \quad \textit{1 was the first to deviate} \\ \varepsilon & \textit{else} \end{cases}$$

This will be an equilibrium if they expect $(A,\varepsilon)_+$ if:

$$\begin{aligned} 4 + \frac{\delta}{1-\delta}4 &\geq 5 + \frac{\delta}{1-\delta}3 \\ \delta &\geq \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 4 + \frac{\delta}{1-\delta}4 &\geq 5 + \frac{\delta}{1-\delta}2 \\ \delta &\geq \frac{1}{3} \end{aligned}$$

So if $\delta \geq \frac{1}{2}$ it is an equilibrium. Like before if they expect some other future then their current action will not affect the future and their current action profile is a NE, so this is an equilibrium as well.

iii. *Find all the pairs of actions that can be supported as equilibria and explain why they can be supported.*
The general strategy is

$$s_{1t} = \begin{cases} c_1 & \text{if} \quad a_{t-1} = c \\ A & \text{if} \quad \text{1 was the first to deviate} \\ C & \text{else} \end{cases} \quad s_{2t} = \begin{cases} c_2 & \text{if} \quad a_{t-1} = c \\ \delta & \text{if} \quad \text{1 was the first to deviate} \\ \varepsilon & \text{else} \end{cases}$$

this can support any payoff that gives strictly more than $(3, 2)$.
This means: $(A, \alpha), (A, \beta), (A, \varepsilon), (B, \alpha), (B, \beta), (D, \alpha), (D, \beta), (E, \psi), (E, \varepsilon)$ can be supported. (B, β), (E, ψ), (A, δ) and (C, ε) can also be supported since they are static Nash equilibria. In total 11 of the 25 payoffs can be supported.

iv. *Consider the strategy:*

$$s_{1t} = \begin{cases} B & \text{if} \quad a_{2t-1} = \alpha \\ C & \text{else} \end{cases} \quad s_{2t} = \begin{cases} \alpha & \text{if} \quad a_{2t-1} = \alpha \\ \varepsilon & \text{else} \end{cases}$$

where a_{2t-1} is the action player 2 chose last period. Show that this is not an equilibrium for any δ, and then explain how it could be altered to be an equilibrium.
Notice first of all that player 1 is best responding, so we will only check player 2's incentives at first. If they expect $(B, \alpha)_+$ then it is an equilibrium when:

$$\begin{aligned} 3 + \frac{\delta}{1-\delta} 3 &\geq 6 + \frac{\delta}{1-\delta} 2 \\ \delta &\geq \frac{3}{4} \end{aligned}$$

However if they expect $(C, \varepsilon)_+$ notice that player 2 can deviate to α and get $(B, \alpha)_+$ as the future. This would not be a best response when:

$$\begin{aligned} 2 + \frac{\delta}{1-\delta} 2 &\geq 1 + \frac{\delta}{1-\delta} 3 \\ \delta &\leq \frac{1}{2} \end{aligned}$$

thus there is no δ that will work. There are several ways to alter this strategy to make it work. For example:

$$s_{1t} = \begin{cases} B & \text{if} \quad a_{t-1} = (B, \alpha) \\ C & \text{else} \end{cases} \quad s_{2t} = \begin{cases} \alpha & \text{if} \quad a_{t-1} = (B, \alpha) \\ \varepsilon & \text{else} \end{cases}$$

or $$s_{1t} = \begin{cases} B & \text{if } \forall t' < t \quad a_{2t'} = \alpha \\ C & \text{else} \end{cases} \quad s_{2t} = \begin{cases} \alpha & \text{if } \forall t' < t \quad a_{2t'} = \alpha \\ \varepsilon & \text{else} \end{cases}$$

10. *Consider the following interaction. A customer can buy food from a restaurant but does not know if it is low or high quality. The restaurant can choose to produce either high (H) or low quality (L) food, and the customer can choose whether to buy (B) or not (N). If the consumer chooses not to buy then both parties get nothing. Assume that the cost of high quality is c_h, the cost of low quality is c_l, and that the value of high quality to the consumer is $v_h > c_h$, and the value of low quality food to the consumer is v_l. Assume that $v_h - c_h > v_l - c_l > 0$ and $c_h > v_l$. Let the price that the food is sold at be p.*

   (a) *Analyzing this as a normal or strategic form game.*

      i. *Letting the customer be the row player, draw a matrix that describes this game.*

	H	L
B	v_h-p,p-c_h	v_l-p,p-c_l
N	0;0	0;0

      ii. *Assuming $p > v_h$ find the best responses of both players and the Nash equilibrium.*

	H	L
B	v_h-p,p-c_h	v_l-p,p-c_l^2
N	$\underline{0;0^{12}}$	$\underline{0;0^{12}}$

      Nash equilibria are underlined, best responses are marked by a 1 or 2 in the upper right hand corner. 1 is the row player.

      iii. *Assuming $v_h > p > c_h$ find the best responses of both players and the Nash equilibrium.*

	H	L
B	v_h-p,p-c_h^1	v_l-p,p-c_l^2
N	0;0^2	$\underline{0;0^{12}}$

      iv. *Assuming $v_l > p > c_l$ find the best responses of both players and the Nash equilibrium.*

	H	L
B	v_h-p,p-c_h^1	$\underline{v_l - p, p - c_l^{12}}$
N	0;0^2	0;0^2

      v. *Is there any common element to the equilibria in all cases? Given the equilibria of these games, what is the profit maximizing choice for p?*
      The best responses of the restaurant do not change with p, if the consumer buys they will always provide low quality. The profit maximizing value for p is v_l. The consumer will still have a best response to buy, and the firm will sell at low quality.

(b) *Analyzing this as an extensive form game.*

i. *Draw an extensive form game with the customer choosing first that describes this interaction.*
I am not willing to go to the work to do this. It is a fairly straightforward exercise.

ii. *Argue that the Weak Sequential equilibria of this game are the same as the Nash equilibria of the Normal form game.*
There is always a WSE in a game, thus if there is one NE then it is that equilibrium. This only leaves the case where $p > v_h$. In this game the customer has a dominant strategy so there action is unique in the sequential game, and given that choice the choice of the firm is transparent.

iii. *Does it matter whether the restaurant knows what the customer has done or not in the extensive form game? Why or why not?*
No it does not, because the restaurant can always play L, it is a best response independent of what the customer does.

(c) *Analyzing this as a repeated game, where the discount factor is δ, $0 < \delta < 1$.*

i. *Show that*

$$\Sigma_{t=0}^{T} \delta^t a = \frac{1 - \delta^{T+1}}{1 - \delta} a \ .$$

You may assume this throughout the rest of the question even if you think your answer is not correct.

$$\begin{aligned}
\Sigma_{t=0}^{T} \delta^t a &= a + \delta a + \delta^2 a + \delta^3 a + ... + \delta^T a \\
(1 - \delta) \Sigma_{t=0}^{T} \delta^t a &= \begin{array}{c} a + \delta a + \delta^2 a + \delta^3 a + ... + \delta^T a \\ -\delta a - \delta^2 a - \delta^3 a - ... - \delta^T a - \delta^{T+1} a \end{array} \\
(1 - \delta) \Sigma_{t=0}^{T} \delta^t a &= a - \delta^{T+1} a \\
\Sigma_{t=0}^{T} \delta^t a &= \frac{1 - \delta^{T+1}}{1 - \delta} a
\end{aligned}$$

ii. *One of your friends claims that he can trust the restaurant to provide high quality because if the restaurant does not then he would never go back.*

A. *Formalize this intuition as a repeated game strategy.*
Solution:

$$s_{ct} = \begin{cases} B & \textit{if} \quad a_{t-1} = (B, H) \\ N & \textit{else} \end{cases}$$

B. *Show that if δ is high enough then the restaurant will produce high quality if consumers use this strategy. Find a formula for how high δ has to be in terms of the other fundamentals of the model. Is it increasing or decreasing in p?*

Solution:
If this is an equilibrium then a best response for the restaurant is:

$$s_{rt} = \begin{cases} H & if \quad a_{t-1} = (B, H) \\ L & else \end{cases}$$

this is a best response when the firm is supposed to choose H if:

$$\begin{aligned} p - c_h + \frac{\delta}{1-\delta}(p - c_h) &\geq p - c_l \\ \delta(p - c_h) &\geq (1-\delta)(c_h - c_l) \\ \delta(p - c_h + (c_h - c_l)) &\geq (c_h - c_l) \\ \delta &\geq \frac{c_h - c_l}{p - c_l} \end{aligned}$$

this is a best response when the firm is supposed to choose L because L is a static best response to N and the future will be unchanged by what he does. The customer is always using a static best response as long as $p \leq v_h$, and if $a_{rt} = H$ then choosing N would decrease the payoff in the future as well, thus the customer is best responding.
Notice that δ is clearly decreasing in p. Since p is a choice variable and δ is a fundamental, notice we can also write this equation as:

$$p \geq \frac{c_h - c_l}{\delta} + c_l$$

C. *Find the profit maximizing value for p when consumers use this strategy (and δ is high enough that the firm will produce high quality). Are their profits higher or lower than in part 10a? Relate this finding to the success of chain restaurants like McDonald's and Burger King.*
Solution: Obviously $p = v_h$. Chain restaurants have an easier time maintaining quality since customers will almost always come back to one of the chain's restaurants. Thus they are successful because they have the incentive to provide high quality.

iii. *Another friend claims that he doesn't have to never go back to that restaurant, if he only doesn't go back for one period it is enough.*

A. *Formalize this intuition as a repeated game strategy.*
Solution:

$$s_{rt} = \begin{cases} N & if \quad a_{t-1} \neq (B, H) \quad and \quad a_{t-2} \neq (N, L) \\ B & else \end{cases}$$

there are many other ways to write this strategy. For example:

$$s_{rt} = \begin{cases} N & if \quad a_{t-1} = (B, L) \\ B & else \end{cases}$$

or

$$s_{rt} = \begin{cases} N & if \quad a_{t-1} \neq a^*_{t-1} \\ B & else \end{cases}$$

where a^*_{t-1} is the action profile that was called for in equilibrium.

B. *Let* $c_l = 0$*,* $c_h = 2$*,* $p = 6$*, and* $\delta = \frac{3}{4}$*, show that the restaurant will produce high quality if consumers use this strategy.*
Solution:

$$\begin{aligned} (p - c_h) + \delta (p - c_h) + \delta^2 v \left((B, H)_+\right) &\geq (p - c_l) + \delta 0 + \delta^2 v \left((B, H)_+\right) \\ (p - c_h) + \delta (p - c_h) &\geq p - c_l \\ \delta &\geq \frac{c_h - c_l}{p - c_h} \end{aligned}$$

$$\frac{3}{4} \geq \frac{2 - 0}{6 - 2} = \frac{1}{2}$$

so it is an equilibrium. In the period when the restaurant expects the consumer not to buy a best response is L, and not playing L will lower his future payoffs, so that is a best response. Given the restaurant's strategy the consumer's actions are always a best response.