ECON 439

Practice Questions, Normal Form Games

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These questions are supposed to help you prepare for exams and quizzes, they are not to be turned in. Answers will be posted before the relevant exam.

1 Chapter 2—Nash Equilibrium, Theory

1. Consider an arbitrary game $G = (I, A, u)$ where I is the finite set of players; A_i is the finite set of strategies for player i, $A = \times_{i \in I} A_i$; and $u_i : A \to R$ is i's payoff function, $u = [u_i]_{i \in I}$. Throughout you may assume players only use pure strategies.

   (a) Define the *rationality* of a strategy in this game.

   (b) Define *correct expectations* in this game.

   (c) Define *best response* in this game.

   (d) Write down two definitions of a (pure strategy) Nash equilibrium in this game. One that uses only best responses, and another one that does not use best responses. Explain why the two definitions are equivalent.

   (e) Explain the relationship between a Nash equilibrium and a social contract, being clear about the attributes of the social contracts you are discussing.

2. Consider the following game:

	α	β	δ	γ
A	$7;2$	$0;2$	$6;4$	$0;2$
B	$3;7$	$14;3$	$8;4$	$-1;4$
C	$0;4$	$13;13$	$3;15$	$1;0$
D	$3;2$	$6;1$	$6;3$	$2;4$

   (a) Find all the best responses of both parties, for **one** person and **one** strategy explain why it is a best response in detail. You may mark your answers on the table above, but explain your notation below.

   (b) Find the pure strategy Nash equilibrium of this game.

   (c) Find the unique dominated strategy in this game. Explain in detail why it is a dominated strategy, and explain carefully why no other strategies are dominated.

   (d) Find the set of strategies that survives iterated deletion of dominated strategies.

   (e) Find the unique mixed strategy Nash equilibrium of this game that does not have any weight on the strategies in the pure strategy Nash equilibrium.

(f) A payoff is *Pareto Efficient* if there is no way to increase one person's payoff without hurting the other persons. Find the set of Pareto Efficient payoffs in this game.

(g) What does this game illustrate about the relationship between Nash equilibria and Pareto Efficiency? Explain why this tension exists.

3. Consider the following Normal form game:

		Player 2		
		α	β	δ
Player 1	A	$6;6$	$6;6$	$0;4$
	B	$5;5$	$2;9$	$16;0$
	C	$3;5$	$-3;4$	$0;3$
	D	$13;8$	$6;6$	$6;3$

(a) Find *all* the best responses for both players, you may mark them on the table above but explain your notation below.

(b) For both players find a dominated strategy, and carefully explain why it is dominated.

(c) After you remove those dominated strategies from the game, find one more dominated strategy and explain which action dominates it.

(d) In this game explain why you should not remove strategies that are weakly dominated.

4. Consider the following Normal form game:

		Player 2			
		α	β	ψ	δ
	A	$10;0$	$6;37$	$4;39$	$0;40$
Player 1	B	$8;3$	$4;6$	$6;7$	$37;4$
	C	$9;4$	$7;6$	$7;5$	$39;3$
	D	$0;10$	$5;9$	$3;8$	$40;0$

(a) Find the best responses for both parties, you may mark them on the table above. For at least one of them **carefully** justify why it is a best response below.

(b) Are there any dominated strategies? If so list them along with what dominates them and **carefully** explain why.

(c) Are there any strategies that can only be eliminated by iterated deletion of dominated strategies? If so list them along with what dominates them and **carefully** explain why.

(d) Is there a Nash equilibrium in pure strategies? If so explain why it is a Nash equilibrium.

(e) Is there a cycle in best responses? If so explain what it is.

(f) Find a mixed strategy Nash equilibrium of this game where actions that are not in a cycle over best responses have zero probability.

5. Consider the following normal form game:

		Player 2 L	C	R
Player 1	U	6; 4	11; 4	1; 5
	M	4; 4	8; 8	0; 6
	D	3; 5	6; 1	2; 4

(a) Find the best responses of both players, **in at least one case explain why it is a best response**. You may mark your answers above but you will loose 2 points if you do not explain your notation below.

(b) There is one dominated action in this game, find it and **carefully show that it is dominated**.

(c) After elimination of that dominated action, there is one new action that is dominated, find it—you do not need to explain why it is dominated in great detail but you do need to tell me which action dominates it.

(d) Find the unique Nash equilibrium of this game.

6. Consider the following strategic form game:

		Player 2 α	β	δ
Player 1	A	1; 2	4; 1	3; 0
	B	3; 3	3; 2	5; 4
	C	4; 5	5; 4	2; 7

(a) Find all the best responses to pure strategies, you may mark them above but explain your notation below.

(b) Show that the unique Nash equilibrium is the only rational action to take in this game. *(Hint: dominated strategies.)*

7. In the following Normal form game:

		Player 2 α	β	ψ	δ	ε
Player 1	A	1; 3	1; 6	6; 5	5; 8	2; 7
	B	3; 2	5; 5	9; 0	5; 1	9; 4
	C	3; 3	8; 9	5; 0	6; 2	6; 10
	D	4; 3	7; 1	6; 0	4; 2	8; 2
	E	2; 9	2; 5	8; 6	7; 8	4; 4

(a) Find all of the best responses, you may mark them in the graph above.

(b) Using iterated removal of dominated strategies, remove two strategies for each person. Explain your work and draw the new game on the next page in the table provided.

Player 2

Strategies of Player 2 $\rightarrow$

Player 1

__;__	__;__	__;__
__;__	__;__	__;__
__;__	__;__	__;__

Strategies of Player 1 $\uparrow$

(c) Find all of the pure strategy Nash equilibria.

(d) Find a cycle in the best responses.

(e) Find a *candidate* for a Nash equilibrium over the cycle you found in the last part of the question. (To be precise, only actions in the cycle have positive probability. By a candidate I mean that if their is a mixed strategy equilibrium over these actions then it must be this candidate.)

(f) Show that the candidate you found in the last part of this question is not actually a Nash equilibrium.

8. Consider the following model of firm location. There are two firms that choose a location at the same time: for $i \in \{a, b\}$, $l_i \in \{1, 2, 3, 4, 5, 6, 7\}$. Each firm's objective is to maximize its number of customers. Each consumer is endowed with a location ($v_k \in \{1, 2, 3, 4, 5, 6, 7\}$ $k \in (1, 2, 3, ..., , 30)$) and go to the firm that is closest to them, choosing each firm with equal likelihood if both firms are equally close. The number of consumers at each location is:

1	2	3	4	5	6	7
6	4	2	2	6	4	6

notice the total number of consumers is 30.

(a) Fill out the following table with the payoffs of firm a from being at location l when firm b is at location $m \in \{1, 2, 3, 4, 5, 6, 7\}$.

Location of firm b:	1	2	3	4	5	6	7
Location of firm a:							.
1		.					
2						.	
3	.						
4							.
5				.			
6		.					
7			.				

(b) Find the equilibrium by iterated removal of strictly dominated strategies.

9. In the following game find the best responses of both players, the Nash equilibrium (or equilibria) and any dominated strategies. You may mark the best responses, but list the Nash equilibrium (or equilibria) and dominated strategies below. For the dominated strategies you must also list the strategies that dominate them.

	α	β	ψ	δ
A	$8,4$	$1,2$	$5,1$	$8,3$
B	$10,1$	$2,2$	$4,0$	$6,0$
C	$9,9$	$0,5$	$12,2$	$4,8$
D	$3,2$	$0,4$	$6,3$	$7,7$

10. Consider the following strategic form game:

		Player 2			
		α	β	χ	δ
	A	$10,8$....	$0,10$....	$3,8$......	$2,8$......
Player 1	B	$1,2$......	$2,6$......	$4,7$......	$5,7$......
	C	$3,0$......	$1,1$......	$0,0$......	$5,0$......
	D	$2,0$......	$4,4$......	$3,8$......	$6,3$......
	E	$2,2$......	$4,4$......	$3,5$......	$7,5$......

(a) find the best responses of both players, you may mark them in the game above or write them down below.

(b) find the Nash equilibria (they are all in pure strategies.)

(c) Write down the definition of a (strictly) dominated strategy.

(d) Remove all strictly dominated strategies and iterate the process until there are no strategies that are strictly dominated in the remaining game. You may mark out the strategies in the game above but indicate the order you remove them and what strategy dominates them below.

(e) Write down the definition of a weakly dominated strategy.

(f) Write down two games that can be derived by removing weakly dominated strategies.

(g) Why is the concept of iterated removal of strictly dominated strategies an implication of rationality? Why is the concept of weakly dominated strategies not an implication of rationality?

2 Chapter 3—Nash Equilibrium, Illustrations.

1. Consider the following game of joint production. Two workers are considering how much to contribute to a project, of which each of them is entitled

half the benefits. Worker i contributes $w_i \geq 0$ to the project, and then if $W = w_1 + w_2$ the benefit of the project is $R(W) = W(44 - 2W)$, and worker i receives $R(W)/2$. The costs of worker 1 are $c_1(w_1, w_2) = w_1^2$, the costs of worker 2 are $c_2(w_1, w_2) = 5w_2^2$.

(a) Set up the objective functions of both workers.

(b) Find their best responses.

(c) Find the Nash equilibrium outputs of both workers.

(d) An outcome is *production efficient* if given the level of output it is not possible to reduce the costs of producing this output. Is the Nash equilibrium production efficient? Why or why not?

(e) An outcome is *Pareto efficient* if it maximizes the sum of the profit of the two workers. Is the Nash equilibrium Pareto efficient? Why or why not? (Note: If you actually find the welfare maximizing work levels they will not be integers.)

2. Consider the classic Bertrand game. Market demand is given by $Q = 64 - 4P$ and there are two firms with the same cost function, $c_i(q_1, q_2) = 2q_i$. Firms compete by choosing price $p_1 \geq 0$ and $p_2 \geq 0$ and for any (p_1, p_2)

$$q_i(p_1, p_2) = \begin{cases} 64 - 4p_i & \text{if } p_i < p_j \\ 32 - 2p_i & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

(a) In a general game, define a *weakly dominated strategy.*

(b) In this game find the Nash equilibrium. (Hint: it is a weakly dominated strategy in this game.)

(c) Now firm 2 advertises that they will always match the price of firm 1. Find the Nash equilibrium of this game given this. (Hint: You don't need to optimize for firm 2 at all.)

(d) Comment on how this illustrates the difference between out of equilibrium logic and equilibrium logic.

3. (Asymmetric Competition) Consider a market where there are two firms producing substitutes. The primary difference in the firms is the strategic variable they use to maximize their profit. Firm b is a Bertrand competitor, and maximizes her profit over price. Firm c is a Cournot competitor, and maximizes his profit over quantity. Their cost functions are $c_1(q_1, q_2) = 4q_1$, $c_2(q_1, q_2) = 4q_2$, or they have the same constant marginal cost. The demand curve for firm B is:

$$q_B = 15 - \frac{1}{2}p_B - \frac{1}{2}q_C \ ,$$

notice I write it in terms of q_C because that is the strategic variable of firm C. The inverse demand curve for firm C is:

$$p_C = 15 + \frac{1}{2}p_B - \frac{3}{2}q_C \ .$$

(a) Show that the demand curves are symmetric. I.e. that they are based on a symmetric underlying model of the relationship between prices and quantities.

(b) Given that the demand curves are symmetric, is this a symmetric problem? Why or why not?

(c) Set up the objective function of both firms.

(d) Find the best responses for both firms.

(e) Find the equilibrium prices and quantities.

(f) Find the profits of both firms. Which firm makes the lower profits?

(g) Now both firms are considering changing their strategic variable, what would you recommend them to switch to? What are the potential problems with changing their strategic variable?

4. Consider the following Hotelling linear city. Customers are distributed over locations the $(1, 2, 3, 4, 5)$, and always buy from the firm that is closest to their location, if two **firms** are an equal distance from them they buy from each firm with equal probability. The distribution of customers is

Locations	1	2	3	4	5
Number of Customers	10	12	14	2	4

firms choose a location(s) to maximize the number of customers it gets, call their payoff π.

(a) The Standard Model: In this model two firms, a and b, both choose a location simultaneously $l_a \in \{1, 2, 3, 4, 5\}$ is a's location and $l_b \in \{1, 2, 3, 4, 5\}$ is b's location.

i. Fill out the table below, in the box I want you to write in the payoff to firm a if firm b is in the location across the top of the table and firm a is in the location on the side. Notice I ONLY want you to write in the payoff of firm a, if you are not clear on what you should do please ask. **Notice that there is a lot of symmetry in this table, if you can figure it out it will make answering it much easier.**

if $l_b =$	1	2	3	4	5
and $l_a = 1$ then $\pi_a =$	___	___	___	___	___
and $l_a = 2$ then $\pi_a =$	___	___	___	___	___
and $l_a = 3$ then $\pi_a =$	___	___	___	___	___
and $l_a = 4$ then $\pi_a =$	___	___	___	___	___
and $l_a = 5$ then $\pi_a =$	___	___	___	___	___

ii. Find the Nash equilibrium by iterated deletion of dominated strategies. **Be careful to make your answers to this question clear. If you mark your last table so much that I can not clearly see what you have written I will mark you down**.

(b) A Big Firm/Small Firm Model: In this model firm a only has one location, l_a, but firm b can choose two locations, l_{b1} and l_{b2}—however (for simplicity) we require that $l_{b1} = l_a$. Note that $l_{b1} = l_{b2} = l_a$ is allowed.

To be clear about how the division of subjects is done, assume that $l_{b1} = l_a = 5$ *and* $l_{b2} = 3$*, then firm b would get all the consumers in locations 1-3 (36) half the consumers in location 5 (2) because* l_{b1} *and* l_a *are equally far from them, and half the consumers in location 4 (1) because* **firms** *a and b are equidistant from them. Notice that firm b will have two branches and firm a will only have one branch equally far from the consumers in location 4, but still each firm gets half the customers.*

i. For each location of firm a find the best responses of firm b. Explain the logic behind your best responses below the table.

if $l_a =$	1	2	3	4	5
The BR of firm b is $l_{b1} =$	1	2	3	4	5
$l_{b2} =$	___	___	___	___	___

ii. For each best response you found above find the optimal location for firm a. Notice that a does not have to choose to be at one of the two locations b has chosen. Explain your logic below the table.

Just to be absolutely clear, let me give an example. Say that $l_{b1} = 1$ *and* $l_{b2} = 4$*, then firm a is free to choose* $l_a = 3$*, and when calculating the payoffs we do not assume that* $l_{b1} = 3$*. In this case* $\pi_a = 20$, $\pi_b = 22$.

if $l_{b1} =$	1	2	3	4	5
and $l_{b2} =$	___	___	___	___	___
$BR_a(l_{b1}, l_{b2})$	___	___	___	___	___

iii. Find the pure strategy Nash equilibria of this game.

iv. Do you think there will always be a pure strategy Nash equilibrium in this model? Why or why not? You can argue either answer, points will be given for the coherence of your argument.

5. In a second price auction there is one item that will be given to one of I bidders. The bidder $i \in \{1, 2, 3, 4, ..., I\}$ has a value for the good $v_i \in [\underline{v}, \overline{v}]$ and submits a bid $b_i \in [\underline{v}, \overline{v}]$ .For simplicity you should assume the bid can only be in kurus, but that $\overline{v} - \underline{v}$ is very large relative to one kurus. Everyone involved in the auction knows everyone else's value, and you can assume without loss of generality that $v_1 > v_2 > v_3 ... > v_I$. A bidder wins if he submits the highest bid. A bidder's payoffs if he wins are $v_i - \max_{j \neq i} b_j$ (he pays the second highest bid); if he ties for the highest bid with K other people his payoff is $\frac{1}{K}(v_i - \max_{j \neq i} b_j)$; and 0 otherwise. Notice that a bidder's own bid never actually determines the price he pays.

(a) Prove that bidding $b_i = v_i$ is a *weakly* dominant strategy.

(b) Prove that if for all i, $b_i = v_i$, then this is a Nash equilibrium.

(c) Prove that if for any $j \in \{1, 2, 3, 4, ..., I\}$ $b_j = \overline{v}$ and for everyone else, $k \in \{1, 2, 3, 4, ..., I\} \setminus j$, $b_k = \underline{v}$ then this is a Nash equilibrium.

6. There are two firms that are Bertrand competitors in a market where market demand is $Q = 56 - 4p$. If they match price, they split demand, otherwise the firm that has the lower price gets all of the demand. Price must be in positive integers (for $i \in \{1, 2\}$, $p_i \in (0, 1, 2, 3, ...)$) and the constant marginal cost of production is 6. Just to be clear, the firm level demand is:

$$d_1(p_1, p_2) = \begin{cases} 0 & p_1 > p_2 \\ \frac{1}{2}(56 - 4p_1) & p_1 = p_2 \\ 56 - 4p_1 & p_1 < p_2 \end{cases}, d_2(p_1, p_2) = \begin{cases} 56 - 4p_2 & p_1 > p_2 \\ \frac{1}{2}(56 - 4p_2) & p_1 = p_2 \\ 0 & p_1 < p_2 \end{cases}$$

(a) Set up the objective function of firm one, you can use the general demand curve $d_1(p_1, p_2)$.

(b) Find the monopoly price, i.e. the price a firm would charge if the other firm did not set a price.

(c) For an arbitrary $p_2 \in (1, 2, 3, ...)$ find the profits for firm one from charging the price $p_2 - 1$, p_2, and $p_2 + 1$. Your answers should all have p_2 in them. (You should find three functions, one for $p_2 + 1$, one for p_2, and one for $p_2 - 1$).

(d) Find the best response of firm one for every $p_2 \in (0, 1, 2, 3, ...)$. Provide a precise mathematical proof when p_2 is equal to

i. the monopoly price plus one,
ii. the monopoly price,
iii. marginal cost plus two,
iv. marginal cost plus one,
v. marginal cost,
vi. marginal cost minus one.

For the rest of the prices you can just generalize the results you found for these prices.

(e) Find the Nash equilibria of this game.

(f) Prove that setting price equal to the constant marginal cost is a weakly dominated strategy. (A strategy is weakly dominated if there is another strategy that always does at least as well and sometimes does strictly better.)

7. Consider the following war of attrition. Two people are fighting over an object which has a value of 1 to the winner. They can fight 0,1,2,3,4,5, or 6 periods, and the cost of a period of fighting is $\frac{3}{4}$ for person 1 and

$\frac{2}{7}$ for person 2. Their strategy is the number of periods they will fight, $t_i \in \{0,1,2,3,4,5,6\}$ where $i \in \{1,2\}$. To be clear their utility function is:

$$u_1(t_1,t_2) = \begin{cases} 1-\frac{3}{4}t_2 & \text{if } t_1 > t_2 \\ \frac{1}{2}\left(1-\frac{3}{4}t_2\right) & \text{if } t_1 = t_2 \\ -\frac{3}{4}t_1 & \text{if } t_1 < t_2 \end{cases}, u_2(t_1,t_2) = \begin{cases} 1-\frac{2}{7}t_1 & \text{if } t_1 < t_2 \\ \frac{1}{2}\left(1-\frac{2}{7}t_1\right) & \text{if } t_1 = t_2 \\ -\frac{2}{7}t_2 & \text{if } t_1 > t_2 \end{cases}$$

(a) Find the best responses for both persons by filling in the table below.

t_2	0	1	2	3	4	5	6
Best Responses of 1							
	....	....	....	....	....	....	....
t_1	0	1	2	3	4		
Best Responses of 2							

(b) Find all of the equilibria. You may either characterize them all or list them one by one.

8. Consider a public good game where each person in the society can contribute $d_i \geq 0$ (to be precise $d_i \in [0,\infty)$), let $D = \Sigma_{i=1}^{I} d_i$ be the total amount contributed. The benefit to each person in the society of D is $B(D) = 80D - 2D^2$, a person gets this benefit no matter how much she or he contributes. For person i, the cost of contributing is $c_i(d_i) = 48d_i$. The objective of each person is to maximize their net benefit, which is defined as the benefit function minus the cost of contributing.

   (a) First assume that there are two people in the society ($I = 2$).

      i. For person one, set up her objective function and find the first order condition.

      ii. Find the best response for one of the people.

      iii. Find the set of dominated strategies in the game. Explain your reasoning.

      iv. Find the amount each person contributes in equilibrium, and the total amount contributed in equilibrium.

   (b) Now assume that there are $I \geq 2$ people in this society.

      i. For person one, set up her objective function and find the first order condition. You may denote the total amount contributed by the rest of the people as D_{-1}.

      ii. Find the best response for one of the people.

      iii. Why can we assume that the amount contributed by all people will be the same in equilibrium?

      iv. Find the amount each person contributes in equilibrium, and the total amount contributed in equilibrium. Your answer should be a function of I, the total number of people in the society. *Be sure to check that if $I = 2$ your answer is the same in parts a and b.*

(c) The social welfare function in this society is the sum of the net benefits of individuals, or the sum of their individual objective functions. I now want you to find the socially optimal amount to contribute for all $I \geq 2$.

   i. Set up the objective function and show that it can be written only in terms of D, the total amount contributed.

   ii. Find the first order condition and the socially optimal total amount to contribute.

   iii. For each I compare the amount you found in part c to the amount you found in part b. Which is higher? Why is this?

9. Consider a Cournot Oligopoly where the inverse demand curve is given by $P = 17 - Q$ and the costs of a type a firm is $c_a\left(q^a, q^b\right) = q^a$ and the costs of a type b firm is $c_b\left(q^a, q^b\right) = 3q^b$.

   (a) In this part of the question assume that there is one firm of both types.

      i. Set up the objective function of both firms.

      ii. Find the best responses of both firms.

      iii. Find the Nash equilibrium quantities.

      iv. Find the profit of both firms in the Nash equilibrium.

   (b) Now assume that there are two firms of type b, firm 1 and firm 2, firm 1 produces q_1^b and firm 2 produces q_2^b.

      i. Set up the objective function of both types of firms.

      ii. Find the best responses of both types of firms.

      iii. Why can you assume that $q_1^b = q_2^b$ in equilibrium?

      iv. Using the insight in the last part of the question, find the Nash equilibrium quantities.

      v. Find the profit of both types of firms in the Nash equilibrium.

   (c) Now assume that the costs of firms of type b is $c_b\left(q^a, q^b\right) = 3q^b + F$, where F is a fixed cost. Further consider a *free entry equilibrium* where as many firms of type b can enter as want. (Note that only one firm of type a will be in the market, and that the costs of that type of firm do not change.) This means that for firms of type b, $\pi_i^b = 0$ in equilibrium.

      i. If $F = 9$, what will be the equilibrium number of firms of type b? Why?

      ii. If $F = 20$, what will be the equilibrium number of firms of type b? What will be total quantity produced by firms in the market?

10. Consider a second price auction. The auctioneer has one item to give to one of I bidders, the bidders each submit a simultaneous bid of $b_i \in [0, 100]$

and the item is rewarded to the highest bidder at the second highest bidder's bid. If more than one bidder is tied for the highest bid the item is awarded to one of them at random. Note that if i is the highest bidder then $\max_{j \neq i} b_j$ is the second highest bid. Each bidder has a value $v_i \in (0, 100)$ and their utility is:

$$u_i(b) = \begin{cases} 0 & \text{if they are not the highest bidder} \\ v_i - \max_{j \neq i} b_j & \text{if they are the unique highest bidder} \\ \frac{1}{J}\left(v_i - \max_{j \neq i} b_j\right) & \text{if there are } J \text{ people who tie for the highest bid.} \end{cases}$$

Assume that $v_1 > v_2 > v_3 > ... > v_I$ and that $I > 5$. Remember that in this game everyone knows everyone else's v_i.

(a) Find a *symmetric* equilibrium strategy and prove that it is an equilibrium.

(b) Find an asymmetric equilibrium where bidder 5 wins the auction and prove that it is an equilibrium.

11. Consider the following market. Demand for two goods is given by:

$$\begin{aligned} q_1 &= 144 - 3p_1 + 2p_2 \\ q_2 &= 144 - 3p_2 + 2p_1 \end{aligned}$$

and the firms have the same constant marginal cost: $c_1(q) = 24q_1$, $c_2(q) = 24q_2$.

(a) First assume that these firms are Bertrand competitors.

i. Set up the objective function of one of the firms.

ii. Find the firm's best response.

iii. Find the equilibrium prices and quantities these two firms charge.

(b) Now assume that these firms are Cournot competitors.

i. Verify that the inverse demand curves are:

$$\begin{aligned} p_1 &= 144 - \frac{2}{5}q_2 - \frac{3}{5}q_1 \\ p_2 &= 144 - \frac{2}{5}q_1 - \frac{3}{5}q_2 \end{aligned}$$

ii. Set up the objective function of one of the firms.

iii. Find the firm's best response.

iv. Find the equilibrium quantities and prices these two firms charge.

12. There are N voters who have positions that can be indicated by the numbers 1 through 7. The number of voter with each position is indicated in the table below:

1	2	3	4	5	6	7
12	0	0	0	4	5	4

(a) What is the average position of these voters? What is the median position of these voters?

(b) Assume that voters always vote for the candidate who's position is closest to their own, and that there are two candidates. When they are indifferent the voters choose each candidate equally likely. Candidates only care about winning the election.

  i. For each position of candidate 2 find the best position (or positions) for candidate 1. Write your answer in the table below:

Position of candidate 2	1	2	3	4	5	6	7
Best positions for Candidate 1	....	....	....	....	....	....	....

  ii. Find the Nash equilibrium of this game.

13. Consider a model of Bertrand competition with differentiated demand. Firm one has a cost function $c_1(q_1) = 24q_1$ and firm 2 has the cost function $c_2(q_2) = 8q_2$. Demand for firm $i \in \{1, 2\}$ is given by (where $j \neq i$):

$$q_i = d_i(p_i, p_j) = 160 - 3p_i + 2p_j$$

(a) find the best response for firm 1 and firm 2 to every price of their opponent. (You may assume that the firm will not shut down.)

(b) find the Nash equilibrium of this game.

14. Consider the three following models of duopoly (two firms competing for profits in the same market.) In all cases firm one has a cost function $c_1(q_1) = 24q_1$ and firm 2 has the cost function $c_2(q_2) = 8q_2$.

(a) The standard Bertrand model. Market demand is given by $D(p)$, where $D(24) > 0$ and the function is continuous and downward sloping. firm i's demand is given by:

$$q_i = d_i(p_i, p_j) = \begin{cases} 0 & \text{if} \quad p_i > p_j \\ \frac{1}{2}D(p_i) & \text{if} \quad p_i = p_j \\ D(p_i) & \text{if} \quad p_i < p_j \end{cases}$$

and price must be in Kurus, which I will denote κ.

  i. find the best response to p_j assuming $(p_j - \kappa)q_i > c_i(q_i)$. (Hint: you should consider $\{p_j + \kappa, p_j, p_j - \kappa\}$ and you can assume that κ is *extremely* small and that p_j is less than the monopoly price for both firms.)

  ii. find a Nash equilibrium of this game. (While there are multiple Nash Equilibria all you have to do is find one and verify that it is an equilibrium.)

  iii. find the profit of the two firms in a Nash equilibria of this game.

(b) Bertrand with differentiated demand. Demand for firm i is given by:

$$q_i = d_i (p_i, p_j) = 160 - 3p_i + 2p_j$$

since it is not helpful in this case price does not have to be in Kurus, (a price like 112.10450327 is fine.)

i. find the best response for firm 1 and firm 2 to every price of their opponent. (You may assume that the firm will not shut down.)

ii. find the Nash equilibrium of this game.

iii. find the quantities of both firms in equilibrium, and write down the profit and simplify as much as possible. (The profit might be too large to calculate easily, do not worry if you can not simplify it completely.)

(c) Hotelling Linear City: There are N consumers located in a line, consumers buy from the firm that is closest to them (if both firms are equi-distant then they choose each firm with probability one half). Price is fixed (above the marginal cost of both firms) and firms compete by choosing location. Thus firms are essentially trying to maximize their demand. The number of consumers at each location is indicated in the table below:

0	1	2	3	4	5	6
3	8	1	3	2	9	9

i. For each location of firm 2 find a best response (location) for firm 1. Write your answer in the table below:

Location of firm 2	0.....	1.....	2.....	3.....	4.....	5.....	6.....
A Best Location for firm 1							
Average Demand of firm 1							

ii. find the Nash equilibrium of this game.

(d) Which of these models is the best model of duopoly? Why is it better than the other ones? Notice that one could argue that any of these models is best, the points will be given for the argument not for "guessing right."

3 Chapter 4—Mixed Strategy Equilibrium

1. Consider the Hawk/Dove game, the players are animals and each animal has two strategies. Either they can be *aggressive* (A) and hunt other animals to eat, or they can be *passive* (P) and eat plants. The payoffs of the game are:

		Player 2: A	Player 2: P
Player 1	A	0; 0	7; 5
	P	5; 7	4; 4

(a) Find the best responses for each player, you may mark them on the game but explain your reasoning carefully below. Half the points will be given for your explanation of your reasoning.

(b) Find the pure strategy Nash equilibria of this game.

(c) An equilibrium is *symmetric* if every agent uses the same strategy in equilibrium. Why might we want to find a symmetric equilibrium in this game?

(d) Carefully show that there is no symmetric pure strategy Nash equilibrium of this game.

(e) Find the unique symmetric Nash equilibrium of this game.

2. About Mixed strategy Nash equilibria.

(a) Define a mixed strategy and explain how you would implement a mixed strategy. In other words if I told you to play an arbitrary mixed strategy what would you need to do?

(b) Define a Mixed strategy Nash equilibrium for a game with a finite number of strategies and a finite number of players.

(c) Assume that player 1 has two strategies, a and b, and that in a mixed strategy Nash equilibrium, σ^*, player 1 is supposed to play both a and b with positive probability. Prove that:

$$u_1\left(a, \sigma^*_{-i}\right) = u_1\left(b, \sigma^*_{-i}\right)$$

where σ^*_{-i} is the equilibrium mixed strategies of the other players. Also explain how this result is useful in finding mixed strategy Nash equilibria.

3. About the definition of Nash equilibrium.

(a) Define a mixed strategy, and a (mixed strategy) best response.

(b) Define a Mixed Strategy Nash equilibrium.

4. Consider a public goods game. There are $I > 1$ people. Each person can decide whether to contribute (C) or not (N). If at least one person chooses C then everyone gets the public good and gets the same benefit of 18, however each person who contributes also has to pay the cost 2. (Thus that person's (or people's) utility is $18 - 2$.) If no one chooses to contribute then everyone gets zero.

(a) In a general I player game find the expected utilities of players from each action. Your formula should be written in terms of random events, like "the probability that the sun is shining." (Hint: think about the mixed strategy payoffs.)

(b) The two person game:

i. Draw a table representing this game in Normal form.
ii. Find the pure strategy best responses of both players and the pure strategy Nash equilibria. You may mark your answers on the table above but you will loose one point if you do not explain your notation below.
iii. These Nash equilibria are all asymmetric, why might we be interested in a symmetric Nash equilibrium for this game?
iv. Find the symmetric (mixed strategy) Nash equilibrium. Let p be the probability that someone chooses C.
v. Find the probability that no one contributes in this mixed strategy Nash equilibrium.

(c) If $I = 3$

i. Find the symmetric mixed strategy Nash equilibrium. Let p be the probability that someone chooses C.
ii. Find the probability that no one contributes in this mixed strategy Nash equilibrium, is it lower or higher than the probability you found in the two player game? Comment on the implications of this.

(d) If $I > 3$ find the symmetric mixed strategy Nash equilibrium. Let p be the probability that someone chooses C. (Hint: You will only be able to find a formula for this answer.)

5. About mixed strategy equilibria:

(a) Define a *mixed strategy equilibrium.* I will give partial credit for any answer that is approximately correct, but for full credit your answer must be precisely correct. *(Note there are several precisely correct answers.)*

(b) Let σ^* be a mixed strategy equilibrium, and σ^*_{-i} be the mixed strategies of the other players in the game, assume that a_i and $\hat{a}_i$ are played with strictly positive probability in σ^*_i, then what do we know about $U_i\left(a_i, \sigma^*_{-i}\right)$ and $U_i\left(\hat{a}_i, \sigma^*_{-i}\right)$? Why?

6. Consider the following normal form game.

		Player 2		
		α	β	ψ
	A	8; 1	8; 0	3; 7
Player 1	B	1; 5	1; 7	2; 6
	C	2; 9	2; 6	6; 7

(a) Find all the best responses to pure strategies. You may mark them above but explain your notation below.

(b) Let p_α be the probability α is played in some mixed strategy, p_β be the probability β is played in the same mixed strategy, and write the payoffs of person 1 given this mixed strategy of player 2.

(c) Let q_A be the probability A is played in some mixed strategy, q_B be the probability B is played in the same mixed strategy, and write the payoffs of person 2 given this mixed strategy of player 1.

(d) Find a cycle in the best responses and explain the cycle below.

(e) Assuming that actions that are not in the cycle you found in the last part of the question have zero probability, find the mixed strategy equilibrium. Afterwards calculate each person's expected utility from playing all of his or her actions in this mixed strategy equilibrium (including the action that is never played. Do not expect the answers to be integers.)

7. In the following Normal form game:

	α	β	ψ	δ	ε
A	1; 2	5; 3	7; 4	4; 2	8; 3
B	0; 6	9; 8	6; 9	8; 10	3; 7
C	1; 13	2; 16	3; 6	0; 9	12; 12
D	2; 5	0; 4	2; 3	1; 4	2; 3
E	1; 4	6; 16	5; 6	10; 10	4; 5

(a) Find all of the best responses, you may mark them in the graph above.

(b) Find all of the pure strategy Nash equilibria.

(c) Find a cycle in the best responses.

(d) Find a Nash equilibrium over the cycle you found in part *c*. (To be precise, only actions in the cycle have positive probability.)

8. Consider the following strategic form game:

		Player 2			
		α	β	χ	δ
	A	6,0......	2,4......	1,2......	1,1......
Player 1	B	2,4......	3,3......	2,6......	2,2......
	C	3,4......	4,3......	1,2......	3,2......
	D	4,4......	6,2......	0,2......	2,2......

(a) find the best responses of both players. You may mark them in the game above or write them down in the space below.

(b) find the Nash equilibrium in pure strategies.

(c) Are there any cycles in best responses? If so mark them in the game above.

(d) find the Mixed strategy Nash equilibrium of this game.

(e) Which equilibrium is better for player 1? Which equilibrium is better for player 2?

4 Chapter 9—Bayesian Games.

1. Consider a seller with a good of unknown worth. The worth of the good he is selling is $w \in \{w_l, w_m, w_h\}$ where $w_l = 1$, $w_m = 12$ and $w_h = 17$. The probability the good is of worth w_h is $\frac{1}{6}$, of worth w_m is $\frac{2}{3}$, and of worth w_l is $\frac{1}{6}$. The value of the good to the potential buyer is $5w$, and the net utility of the buyer is $u_b(w) = 5w - p$ and of the seller is $u_s(w) = p - w$.

   The seller makes an offer to sell the good at a price p, and the buyer can either accept the offer or reject it. Like usual we assume the buyer will accept the offer if he is indifferent between accepting and rejecting.

   (a) Prove that if the buyer expects the good to be from the set $X \subseteq \{w_l, w_m, w_h\}$ that the equilibrium price will be $p = 5E[w|X]$.

   (b) Prove that there is a *market collapse* equilibrium, or an equilibrium where $X = \{w_l\}$.

   (c) Find the other equilibrium, and be sure to prove that there is only one other equilibrium.

   Now we will consider **certification** in this market. This means that for the fee of 45 the seller can have the worth of his good revealed to the buyer.

   (d) Prove that there is an equilibrium where sellers with high or medium quality goods certify the worth of their good.

   (e) Pareto rank the equilibria you have found (with and without certification), being sure to consider both the buyer's and the seller's preferences.

2. Consider a Differentiated Demand Bertrand Oligopoly where firm 2 is uncertain about firm 1's costs. Both firms know firm 2's costs, they are $c_2(q_2) = 6q_2$. Firm 1 know's firm 1's costs, but firm 2 thinks they are high (h) $(c_1(q_1) = 12q_1)$ with probability $\phi \in (0,1)$ and they are low (l) $(c_1(q_1) = 0)$ with probability $1 - \phi$. The demand for the two firms is:

$$\begin{aligned} q_1 &= 4 - \frac{2}{3}p_1 + \frac{2}{3}p_2 \\ q_2 &= 4 - \frac{2}{3}p_2 + \frac{2}{3}p_1 \end{aligned}$$

   and just to be clear in this model firm's maximize their profits by choosing their price.

   (a) The simultaneous game: Assume both firm's choose their prices at the same time.

      i. Set up the three relevant objective functions in this model.

      ii. Find the three best responses.

      iii. Find the equilibrium prices of both firms.

iv. Find the profits of firm one when her costs are low $(c_1(q_1) = 0)$. You may not want to expand this out completely. You should find p_{1l} and q_{1l} but you may not want to do the final multiplication. *(Hint: quantity may not be an integer.).*

(b) The sequential game: Now firm 1 chooses her price first, and then firm 2 chooses his price.

i. Explain why you do not need to solve for firm 2's best response again.

ii. Set up the two objective functions of firm 1.

iii. Find the equilibrium prices of both firms. Be sure to consider all cases carefully.

iv. Find the profit of firm 1 when her costs are low $(c_1(q_1) = 0)$. *(Hint: quantity should be an integer.)*

(c) Show that if $\phi = 1$ then the profits of firm 1 when her costs are low are higher in the simultaneous game than in the sequential game. Argue by continuity that this should also be true for ϕ close to one.

(d) The principle of First Mover's Advantage says that the first mover in a sequential game should always make higher profits than she makes in the simultaneous game. You have just shown this is sometimes not true in this game. Explain why this principle is violated. *(Note you can assume the answer to the last part of the question here, indeed you don't need to solve any of the rest of the question to answer this question. This is purely about explaining why the principle is wrong.)*

3. Consider the Hawk/Dove game, the players are animals and each animal has two strategies. Either they can be *aggressive* (A) and hunt other animals to eat, or they can be *passive* (P) and eat plants. Unlike last time each party now faces uncertainty about the benefit of the other from being aggressive. Now $U_1(A, P) = b_1$ and $U_2(P, A) = b_2$ or the payoffs of the stage game are:

		Player 2: A	Player 2: P
Player 1	A	$0; 0$	$b_1; 1$
	P	$1; b_2$	$2; 2$

Person i always knows b_i, all that person i knows about person j's b_j is that it is distributed uniformly over $[0, 30]$. Notice that $F(b) = \Pr(b_j \leq b) = \frac{b}{30}$.

(a) For the first two questions treat this as a game of complete information. (person 1 knows b_2 and person 2 knows b_1)

i. Find the best responses and Nash equilibria for **all** (b_1, b_2). Please do not write the best responses on the game above unless it is the same for all values of (b_1, b_2).

ii. Find a mixed strategy Nash equilibrium of this game. *Hint: It may actually be in pure strategies, but you need to prove that. And notice that it will not generally be symmetric.*

(b) Now consider the original Bayesian game, to be precise person i knows b_i but all they know about b_j is that it has the distribution given above.

i. Define a *cutoff strategy* and formulate a reasonable cutoff strategy for this game,

ii. Given the cutoff strategies you formulated in the last part find the expected utility of each action.

iii. Now find the Bayesian Nash equilibrium of this game if everyone uses symmetric cutoff strategies. *Hint: For the quadratic function* $ax^2+bx+c=0$, $x \in \left\{\frac{1}{2a}\sqrt{b^2-4ac}-\frac{1}{2a}b, -\frac{1}{2a}\sqrt{b^2-4ac}-\frac{1}{2a}b\right\}$, *and you should be interested in the larger root.)*

4. Consider a standard auction with imperfect information. A bidder knows his own value v_i—$i \in \{1,2,3,...,I\}$ but all he knows about other bidder's values are that each one is distributed uniformly over $[0,1]$, so the cumulative distribution function for bidder j's value ($j \neq i$) is $F(v_j) = v_j$. The winner will always be the person who bid the highest, and if person i wins and has to pay p then their utility function is $v_i - p$. If person i does not win they get zero. If several people bid the same amount they are equally likely to win.

(a) Second Price Auction: In this auction the high bidder has to pay the second highest bid.

i. Show that bidding your own value, $b_i = v_i$, is a weakly dominant strategy in this game.

ii. Show that there is an equilibrium where $b_4 = 1$ and $b_j = 0$ for every $j \in \{1,2,3,...,I\} \backslash 4$.

(b) The First Price Auction: In this auction the high bidder pays the amount they bid. Assume throughout that they use a symmetric strategy of the form $b_i = \alpha v_i + \beta$, where $\alpha > 0$.

i. Write down the objective function of a bidder in this auction.

ii. Prove that if $v_i = 0$ then $b_i = 0$, or that $\beta = 0$.

iii. Find the first order condition of the objective function. (Assume that $\beta = 0$.)

iv. Find the formula for the bid, and verify that it has the linear form $b_i = \alpha v_i$.

5. Consider a Bertrand game of differentiated demand. The demand for firm 1 and 2 is:

$$q_1 = 54 - p_1 + \frac{1}{2}p_2$$

$$q_2 = 54 - p_2 + \frac{1}{2}p_1$$

The costs of firm 2 are $c_2(q_1, q_2) = 0$, the costs of firm 1 are $c_1(q_1, q_2) = 0$ with probability ρ and $c_1(q_1, q_2) = 60q_1$ with probability $1 - \rho$, where $\rho \in (0, 1)$. Firm 1 knows her costs, firm two does not.

(a) Set up the two objective functions for firm 1.

(b) Find the two best response formulas for firm 1.

(c) Find the expectation of p_1 for firm 2, it should be a function of p_2.

(d) Set up the objective function for firm 2, be sure to include the fact that p_1 is a random variable.

(e) Find best response formula for firm 2.

(f) Find the Bayesian Nash equilibrium prices.

6. Cafe Nero has finally come to Ankara! And being the sensible people they are they decided to open near to Bilkent first, specifically in the Real shopping center. Unfortunately this causes you a problem, you usually go to flirt with that special someone at Starbucks, and now you are afraid that you might want to go to Cafe Nero instead.

   Your strategy set is N—go to Care Nero—and S—go to Starbucks. Both of you get a utility of 1 from being at the same coffee shop with that other person and 2 from going to Starbucks, however your utility of going to Cafe Nero is unknown. Each of you knows your own u_i, but all you know about the other person's u_j is that it is uniformly distributed over $[0, 7]$, thus it has the cumulative distribution function of $F(u_j) = \frac{u_j}{7}$ $(j \neq i)$.

   (a) Draw a normal form game that represents this situation. You should have the values u_1 and u_2 in the payoffs of this table.

   (b) Prove that there is no pure strategy Nash equilibrium of this game, i.e. you can not choose N all the time and you can not choose S all the time. (You should consider different values of u_i and u_j).

   (c) Write down a cut off strategy that you may want to use in this game.

   (d) Given this cutoff strategy write down the expected payoffs of using the strategies N and S.

   (e) Find the equilibrium cut off strategies in this game, you may assume they are symmetric.

7. Consider two firms that are simultaneously deciding whether or not to enter an industry. If they stay out (O) they get zero. If they enter (E) they have to pay a fixed cost f_i $(i \in \{1, 2\})$. This cost is private information for firm $i \in \{1, 2\}$, all the other firm knows is that it is distributed uniformly over $[0, 12]$. The cumulative distribution function of f is $G(f) = \frac{f}{a}$. If only one firm enters it earns monopoly revenue of 8 so it's total profit is $8 - f_i$, if both firms enter then they earn duopoly revenue of 4, so their total profits are $4 - f_i$.

(a) Write down a normal form game with the payoffs above, note that f_i will vary and should be a part of your payoffs.

(b) Prove that there is no pure strategy equilibrium in this normal form game. Where by pure strategy I mean that one person takes the same action for all f_i.

(c) What is a cutoff strategy? What type of cutoff strategy do you think people will use in this game?

(d) Find a symmetric equilibrium in cutoff strategies.

8. Consider the following public goods game. There are three people in this society. Donating costs $c \geq 0$, if one person donates then the public good is produced, giving each person a benefit of 4. The utility of a person is their benefits minus their costs. If someone donates this is denoted D, if she or he does not this is denoted N.

   (a) Assume that $0 < c < 4$ and that c is common knowledge and the same for all parties. Some answers will depend on c.

      i. Find the pure strategy best responses.

      ii. Find the pure strategy Nash equilibria.

      iii. Notice that none of these pure strategy Nash equilibria are *symmetric*, why might we be interested in a symmetric Nash equilibrium?

      iv. Find a symmetric Nash equilibrium, and then calculate the probability that the public good will be provided in that symmetric Nash equilibrium. *(Hint: this probability is not one.)*

   (b) Assume that c is distributed over $[0, 16]$ with the cumulative distribution function $F(c) = \frac{1}{4}\sqrt{c}$; and that each person knows her or his personal value of c, but not the other players.

      i. Prove that there is no pure strategy equilibrium.

      ii. Find a symmetric equilibrium in cutoff strategies.

      iii. Find the probability that the public good will be provided in this equilibrium.

9. Consider a market for Bertrand with differentiated demand. The demand curves of the firms are symmetric:

$$\begin{aligned} q_1 &= 108 - 2p_1 + p_2 \\ q_2 &= 108 - 2p_2 + p_1. \end{aligned}$$

However while firm 1 has the cost function of $c_1(q_1) = 15q_1$ all it knows about firm 2's cost is that it is $c_2(q_2) = 20q_2$ with probability ρ and $c_2(q_2) = 0$ with probability $1 - \rho$. Firm 2 knows its own costs and the costs of firm 1.

(a) Set up all the objective functions for the two firms.

(b) Find the best responses for all types of all firms.

(c) Find the price firm 1 chooses in equilibrium.

(d) Find the prices firm 2 may choose in equilibrium.

10. Consider the Prisoner's Dilemma with costs of betrayal. Two criminals (and friends) are caught committing a robbery. During the robbery a murder was committed but the police have no evidence the criminals committed it.

The police tell each criminal that if only one of them confesses (C) to the murder then that person will go free, with no prison sentence, the other will be convicted of the murder and the robbery. If both confess then both will be convicted of murder. If neither confess (both choose quiet, Q) then both of them will be convicted for the robbery.

The difference between this and the standard prisoner's dilemma is that now a criminal feels bad if he confesses. Confessing will cost that criminal c_i. This is private information to person $i \in \{1, 2\}$, all that the other person knows is that it is distributed uniformly over $[0, 5]$, with a cumulative distribution function of $F(c) = \frac{c}{5}$. The payoffs in the stage game are the number of years spent in prison minus the cost of confessing (if applicable).

		Player 2: C	Player 2: Q
Player 1	C	$-20 - c_1; -20 - c_2$	$-c_1; -22$
	Q	$-22; -c_2$	$-2; -2$

assume throughout that both players will use a cutoff strategy.

(a) Find the expected payoff of player 1 from playing C and Q for any cutoff strategy of player 2.

(b) Prove that there is no equilibrium where both parties always choose one of the two strategies.

(c) Find all the equilibria where both parties sometimes confess.

11. Consider the following two normal form games, in this question only analyze pure strategies.

Game α

Player 1 \ Player 2	L	C	R
U	$5, 3$	$0, 4$	$6, 5$
M	$4, 1$	$5, 3$	$7, 0$
D	$3, 1$	$0, 0$	$0, 2$

Game β

Player 1 \ Player 2	O	T	A
U	$6, 5$	$6, 3$	$0, 2$
M	$0, 1$	$10, 0$	$5, 2$
D	$7, 0$	$4, 1$	$0, 2$

(a) Find the best responses and Nash equilibria in both games. You may mark the best responses on the graph above. Write down the equilibrium *strategies* below.

(b) Now assume that player 1 does not know which game he is playing, instead he thinks he is playing game α with probability p and game β with probability $1-p$.

i. For each strategy of player 1, $s_1 \in \{U, M, D\}$, find the best response of player 2.

ii. Explain why if we want to find a pure strategy Nash equilibrium we can ignore player 2's strategies that are not best responses to some $s_1 \in \{U, M, D\}$.

iii. For each of the strategies of player 2 found in part *b.i.* of this question and all values of p find the expected utility of player 1 of playing each action against that strategy in the table below. Across the top you should write down each of the three strategies you found in the last part of the question, and then below write the expected value of each action against that strategy.

If $S_2 =$			
$U_1(T, S_2)$			
$U_1(M, S_2)$			
$U_1(B, S_2)$			
	....	....	....

iv. For all values of p find the Nash equilibria.

12. Assume that two firms are Bertrand competitors with differentiated products. Each firm's demand curves are:

$$q_1 = 60 - p_1 + \frac{2}{3}p_2$$
$$q_2 = 60 - p_2 + \frac{2}{3}p_1$$

firm 2's costs are $c_2(q) = 32pq_2$, firm 1's costs are $c_1(q) = 0$ with probability p and $c_1(q) = 96q_1$ with probability $1-p$.

(a) Find the best response of firm 1 to firm 2's price if $c_1(q) = 96q_1$.

(b) Find the best response of firm 1 to firm 2's price if $c_1(q) = 0$.

(c) Find the best response of firm 2 to firm 1's price.

(d) Find the equilibrium prices of the two firms. *Hint—there is something peculiar about the equilibrium.*

13. There are two firms that are both working to invent the *brain chip*, which will allow people to type into computers by only thinking about it. If company $i \in \{1, 2\}$ invents the brain chip it will cost them C_i, which is known

only to that company. They simultaneously decide whether to *invent* (I) or *not invent* (N). If they choose I they invent and sell the brain chip and it costs them C_i. All that the other firm and the government knows about C_i is that it is distributed independently and uniformly over $[0, 60]$, this means that $F(C) = \Pr(C_i \leq C) = \frac{C}{60}$. Each firm knows how much it will cost them to invent the brain chip, the government does not. The government is deciding whether or not to grant the inventor a monopoly (patent) on the brain chip.

(a) Assume that the government does not grant a monopoly to the inventor. Then if either one invents the brain chip both companies will get 12 because they will both produce the brain chip and sell it.

i. Prove that there can be no equilibrium where one firm always invents and the other does not.

ii. Find the payoff of a representative firm if they invent the brain chip.

iii. Find the payoff to a representative firm if they do not invent the brain chip.

iv. Find the symmetric equilibrium.

(b) Assume that the government does grant a monopoly to the inventor. Then if one firm invents it that firm will get 48, if both firms invent it then both firms will get 12. Notice that since firms decide simultaneously whether to invent it or not it is possible for them both to invent it at the same time.

i. Prove that there can be no equilibrium where one firm always invents and the other does not.

ii. Find the payoff of a representative firm if they invent the brain chip.

iii. Find the payoff to a representative firm if they do not invent the brain chip.

iv. Find the symmetric equilibrium.

(c) Assume that the government only cares about the probability that the brain chip is invented. Find this probability for each case (a and b) and find out whether the government should issue monopolies to inventors (patents) or not.

14. Consider the following two normal form games, in this question only analyze pure strategies.

Game α

		Player 2: L	Player 2: R
Player 1	T	9; 3	0; 5
	M	0; 1	3; 2
	B	6; 8	2; 4

Game β

		Player 2: O	Player 2: A
Player 1	T	3; 2	0; 1
	M	0; 6	9; 4
	B	2; 7	6; 8

(a) Find the best responses and Nash equilibria in both games. You may mark the best responses on the graph above.

(b) Now assume that player 1 does not know which game he is playing, instead he thinks he is playing game α with probability p and game β with probability $1 - p$.

i. For all values of p find the payoffs of player 1 and fill out the following table :

If $S_2 =$	(L, O)	(L, A)	(R, O)	(R, A)
$U_1(T, S_2)$				
$U_1(M, S_2)$				
$U_1(B, S_2)$				
	....	....	....	....

ii. For all values of p find the Nash equilibria.

(c) For which values of p do both players prefer that player 1 does not know what game he is playing?

15. Consider a second-price auction with a binding is reservation price r.

In a second price auction there are I bidders who have values identically and independently distributed on $[\underline{v}, \bar{v}]$, the person who bids the most wins the item and pays the second highest bid. If i has the value v_i and wins at the price of p her utility is $v_i - p$, otherwise it is zero.

If there is a reservation price then the winner must always pay at least r, and they must bid more than r to win. We say that r is *binding* if there is a strictly positive probability that any bidder's value is strictly lower than r.

Prove that if there is a binding reservation price then there is no equilibrium where a given person (1 for example) *always* wins the auction, regardless of the values of the bidders.

16. A consumer is buying a car of unknown value. He knows that the car is equally likely to be worth $k * 1000$ for $k \in \{1, 2, 3, 4, 5, 6, 7\}$. If the car is worth $k * 1000$ to the seller then the buyer values the car at $\beta * k * 1000$. The seller knows the value of the car. The buyer makes a take it or leave it offer.

(a) Given that the buyer offers p, which sellers will be willing to sell their car?

(b) Given that the buyer offers p, what is the average value of the car the buyer will receive?

(c) For **all** values of β find the optimal amount for the buyer to offer.

(d) Why is this called a model of *adverse selection*?

17. Assume that there is a monopolist who makes a take it or leave it offer to a consumer. The monopolist has no value to the good, the buyer has a value which is distributed on the interval $[v_l, v_h]$ with a CDF of $F(\cdot)$ and a PDF of $f(\cdot)$.

    (a) Assume first of all that the probability of v_l is p and the probability of v_h is $1-p$.

        i. Prove that the monopolist will never make an offer that is neither v_l nor v_h,

        ii. Find the critical value of p such that the monopolist will make an offer of v_l.

        iii. Consider changing $v_l + b$ and v_h to $v_h + b$, find the critical value of p such that the offer is v_l as a function of b.

        iv. Prove that it is always Pareto efficient for the monopolist to sell the good to all types of consumers.

    (b) Now characterize the optimal price for the monopolist for general $F(\cdot)$. (This should be found in terms of the first order condition.)

18. Consider a market where the value of the object is known to the seller but not to the buyer. If the object has a value of w to the seller than it has a value of $v_i w$ to buyer i. Assume that there are a large number of buyers and sellers so the price will be determined so that the demand equals the expected supply.

    The value to the seller is distributed uniformly over the range $[0, 10]$, and there are 100 sellers, each of whom has one unit to sell. Quantity is divisible.

    If Q units are sold on the market then the marginal unit will be sold to someone for whom $v_i = 252 - 5Q$.

    (a) Given that the market price is P, find the expected value of the marginal buyer.

    (b) Given that the market price is P, find the expected supply.

    (c) Find the equilibrium quantity that will be sold in this market and the price at which it will be sold.

19. Consider a bank which is lending to investors. All investors need 1000 YTL, and they will have a return of 5 YTL per lira invested with probability γ, and 0 with probability $1-\gamma$. Find a condition on γ such that a bank can afford to offer loans to the investors.

20. Consider a model of Firm-Union bargaining. The revenue the firm will generate in the next year is 14, and this is known to all parties. However the firm does not know whether the union is strong or weak. If the union is strong and is offered any wage below 8 it will go on strike and both

parties will get a payoff of zero. If it is weak then it will accept any offer above 2, if it is offered a lower wage then it will go on strike. The probability that the union is strong is q. *Assume that both parties will always accept any offer if they are indifferent between accepting it and rejecting it.*

(a) First consider a model where the firm makes a take it or leave it offer of w, and the union can either accept it or go on strike.

  i. Find the best response of both types of unions to a wage offer of w.
  ii. Write down the profit of the firm when they offer 8 and when they offer 2.
  iii. If $q = \frac{1}{4}$ what wage will the firm offer? Will the union go on strike?
  iv. If $q = \frac{3}{4}$what wage will the firm offer? Will the union ever go on strike?
  v. Someone points out that the fact that the union goes on strike proves that is strong, and therefore the firm should offer anyone who goes on strike a high wage. This does not work, why not?

(b) Now consider a model where both the firm and the union simultaneously declare a wage. If the wage offered by the firm is higher than the wage offered by the union then the union gets the wage the firm offers, otherwise the firm goes on strike.

  i. Find an equilibrium where the firm never strikes.
  ii. Show that the best equilibrium for the firm when $q = \frac{1}{4}$ and $q = \frac{3}{4}$ are as you found above in parts a.iii and a.iv.
  iii. Describe the full set of equilibria when $q = \frac{1}{4}$ and $q = \frac{3}{4}$.

21. Consider a second-price auction. In this auction there is one indivisible good that is awarded to each of the highest bidders with equal likelihood (notice if there is only one high bidder then it is given to that bidder with certainty). Each bidder submits one sealed bid and the price the highest bidder pays is the highest of all the other bids. The values of the bidders are distributed on $[v_l, v_h]$ where $\infty > v_h > v_l > 0$. For simplicity assume that bids must be in Kurus, that value of every bidder is in Kurus, and that one Kurus is very small relative to the value of each bidder (or v_l).

(a) Prove that it is weakly dominant for a bidder to bid his value. (In other words always an optimal strategy regardless of the strategies of the other players.)

(b) Find an equilibrium where bidder 1 always wins.

(c) In many auctions there is a *reservation price*, or a price at which the object is only sold when the highest bid is (weakly) higher than the reservation price, and the price is always at least the reservation price. We say that a reservation price (r) is *binding* if $F(r) > 0$, or there is a positive probability that the value of any bidder is strictly below this price, and thus also a positive probability that the value of every bidder is below this price.

Show that if there is a binding reservation price then there is no equilibrium where bidder 1 always wins if his value is higher than the reserve price. *(Hint: Think about the cases where* $v_1 < r$ *and* $v_1 = r + \varepsilon$*).*